I. BERRY CONNECTION EFFECTS

A. Relation of the Berry curvature to spin winding

Several of you have wondered about the explicit relatioship, which I advertized but never proven or even shown really, beteween the berry phase flux and the winding of the spin. Here is then the proof.

Let's assume that we have two parameters. k_x and k_y, that parametrize the spn of a spin-1/2 state in a closed manifold. I'm not quite sure why these parameters were exactly chosen. They just came to me somehow from midair. I really don't know... The Berry curvature flux is:

$$\int dk_y \int dk_x \Omega_{\vec{k}} = i \int dk_y \int dk_x \sum_{i,j=x,y} \epsilon^{ij} \frac{\partial}{\partial k_i} \Lambda_j \tag{1}$$

with $\Lambda_j = i \left\langle \vec{k} \right| \frac{\partial}{\partial k_j} \left| \vec{k} \right\rangle$, and $\epsilon^{xx} = \epsilon^{yy} = 0$, $\epsilon^{xy} = -\epsilon^{yx} = 1$. Clearly, the first derivative must fall on the $\left\langle \vec{k} \right|$, otherwise, the antisymmetric tensor will kill the double derivative:

$$\sum_{i,j=x,y} \epsilon^{ij} \left\langle \vec{k} \right| \frac{\partial^2}{\partial k_i \partial k_j} \left| \vec{k} \right\rangle = 0. \tag{2}$$

Therefore:

$$\int dk_y \int dk_x \Omega_{\vec{k}} = i \int dk_y \int dk_x \sum_{i,j=x,y} \epsilon^{ij} \frac{\partial}{\partial k_i} \left\langle \vec{k} \right| \frac{\partial}{\partial k_j} \left| \vec{k} \right\rangle \tag{3}$$

Now we need a leap of faith. Let's prove that we can upgrade the expression above to:

$$\int dk_y \int dk_x \Omega_{\vec{k}} = i \int dk_y \int dk_x \sum_{i,j=x,y} \epsilon^{ij} \mathrm{Tr} \left(\frac{\partial}{\partial k_i} \left(\left| \vec{k} \right\rangle \left\langle \vec{k} \right| \right) \frac{\partial}{\partial k_j} \left(\left| \vec{k} \right\rangle \left\langle \vec{k} \right| \right) \left| \vec{k} \right\rangle \left\langle \vec{k} \right| \right) \tag{4}$$

It is easy to prove this using the antisymmetric matrix. Any term that has a product of $\Lambda_i \Lambda_j$ dies upon summation.

To see this explicitly, we can do the following:

$$\begin{aligned} \int dk_y \int dk_x \sum_{i,j=x,y} \epsilon^{ij} \tfrac{\partial}{\partial k_i} \left\langle \vec{k} \right| \tfrac{\partial}{\partial k_j} \left| \vec{k} \right\rangle &= \int dk_y \int dk_x \sum_{i,j=x,y} \epsilon^{ij} \left\langle \vec{k} \right| \left[\tfrac{\partial}{\partial k_i} \left(\left| \vec{k} \right\rangle \left\langle \vec{k} \right| \right) - \tfrac{\partial}{\partial k_i} \left| \vec{k} \right\rangle \right] \left\langle \vec{k} \right| \tfrac{\partial}{\partial k_j} \left| \vec{k} \right\rangle \\ &= \int dk_y \int dk_x \sum_{i,j=x,y} \epsilon^{ij} \left\langle \vec{k} \right| \tfrac{\partial}{\partial k_i} \left(\left| \vec{k} \right\rangle \left\langle \vec{k} \right| \right) \tfrac{\partial}{\partial k_j} \left| \vec{k} \right\rangle \end{aligned} \tag{5}$$

The last equality follows since:

$$\sum_{i,j=x,y} \epsilon^{ij} \left\langle \vec{k} \right| \frac{\partial}{\partial k_i} \left| \vec{k} \right\rangle \left\langle \vec{k} \right| \frac{\partial}{\partial k_j} \left| \vec{k} \right\rangle = 0 \tag{6}$$

Next, we need to defining the projector $P_{\vec{k}} = \left| \vec{k} \right\rangle \left\langle \vec{k} \right| = \frac{1}{2} + \frac{1}{2} \hat{n}_{\vec{k}} \cdot \vec{\sigma}$ which projects on the state $\left| \vec{k} \right\rangle$, using:

$$\hat{n}_{\vec{k}} \cdot \vec{\sigma} \left| \vec{k} \right\rangle = \left| \vec{k} \right\rangle. \tag{7}$$

Using this simplifies the expression a bit, and we proceed to modify the second derivative in the expression:

$$\begin{aligned} & i \int dk_y \int dk_x \sum_{i,j=x,y} \epsilon^{ij} \left\langle \vec{k} \right| \tfrac{\partial P_{\vec{k}}}{\partial k_i} \tfrac{\partial}{\partial k_j} \left| \vec{k} \right\rangle \\ &= i \int dk_y \int dk_x \sum_{i,j=x,y} \epsilon^{ij} \left\langle \vec{k} \right| \tfrac{\partial P_{\vec{k}}}{\partial k_i} \left[\tfrac{\partial}{\partial k_j} \left(\left| \vec{k} \right\rangle \left\langle \vec{k} \right| \right) - \left| \vec{k} \right\rangle \tfrac{\partial}{\partial k_j} \left\langle \vec{k} \right| \right] \left| \vec{k} \right\rangle \end{aligned} \tag{8}$$

Now:

$$\begin{aligned} & \sum_{i,j=x,y} \epsilon^{ij} \left\langle \vec{k} \right| \tfrac{\partial P_{\vec{k}}}{\partial k_i} \left| \vec{k} \right\rangle \left(\tfrac{\partial}{\partial k_j} \left\langle \vec{k} \right| \right) \left| \vec{k} \right\rangle \\ &= \sum_{i,j=x,y} \epsilon^{ij} \left[\tfrac{\partial \left\langle \vec{k} \right|}{\partial k_i} \left| \vec{k} \right\rangle \tfrac{\partial \left\langle \vec{k} \right|}{\partial k_j} \left| \vec{k} \right\rangle - \left\langle \vec{k} \right| \tfrac{\partial \left| \vec{k} \right\rangle}{\partial k_i} \left\langle \vec{k} \right| \tfrac{\partial \left| \vec{k} \right\rangle}{\partial k_j} \right] = 0 \end{aligned} \tag{9}$$

So the correction term in Eq. (8) can be ignored. It vanishes. Then picking up from Eq. (8), we find:

$$\int dk_y \int dk_x \Omega_{\vec{k}} = i \int dk_y \int dk_x \sum_{i,j=x,y} \epsilon^{ij} \mathrm{Tr}\left[\frac{\partial P_{\vec{k}}}{\partial k_i}\frac{\partial P_{\vec{k}}}{\partial k_j}\left|\vec{k}\right\rangle\left\langle\vec{k}\right|\right] = i \int dk_y \int dk_x \sum_{i,j=x,y} \mathrm{Tr} P_{\vec{k}}[\frac{\partial}{\partial k_x}P_{\vec{k}}, \frac{\partial}{\partial k_y}P_{\vec{k}}] \quad (10)$$

The only contributing terms in the commutator are the ones with the unit vector. These yield a pali matrix. Unless they find another pauli matrix, the trace would kill them. So this reduces further to:

$$\int dk_y \int dk_x \Omega_{\vec{k}} = \frac{i}{8} \int dk_y \int dk_x \sum_{i,j=x,y} \mathrm{Tr}\left(\hat{n}_{\vec{k}}\cdot\vec{\sigma}\left[\frac{\partial \hat{n}_{\vec{k}}}{\partial k_x}\cdot\vec{\sigma}, \frac{\partial \hat{n}_{\vec{k}}}{\partial k_y}\cdot\vec{\sigma}\right]\right). \quad (11)$$

And the only waythat this is nonzero, is if the pauli matrices form a triple product:

$$\mathrm{Tr}(\sigma^x[\sigma^y, \sigma^z]) = 4i. \quad (12)$$

This reduces the integral to a simple vector identity:

$$\int dk_y \int dk_x \Omega_{\vec{k}} = -\frac{1}{2} \int dk_y \int dk_x \sum_{i,j=x,y} \hat{n}_{\vec{k}} \cdot \left(\frac{\partial \hat{n}_{\vec{k}}}{\partial k_x} \times \frac{\partial \hat{n}_{\vec{k}}}{\partial k_y}\right) \quad (13)$$

This is nothing but half the Poyntriagin integral which tells you how much of the unit sphere is explored by the unit vector $\hat{n}_{\vec{k}}$.

B. Jackiw-Rebbi mode

The Thouless pump is like a prototype of a topological phase. We'll learn quickly that topological phases are often synonymous with having an edge state protected by symmetry. This edge state is already obvious from the Thouless example.

What happens if the phase of the potential changes from being $\phi = 0$, $x < 0$ to $\phi = \pi$, $x > 0$. This implies a change of sign of g:

$$\hat{\mathcal{H}}_k = vk\sigma^z + |g| sign(x)\sigma^x \quad (14)$$

If we draw this potential, it'll like a missing step. For this potential this will imply a zero mode. Let's derive it. First, write the hamiltonian fully in real space:

$$\hat{\mathcal{H}} = v\frac{1}{i}\frac{\partial}{\partial x}\sigma^z + |g| sign(x)\sigma^x \quad (15)$$

First, how can I be so confident that it is a zero mode? Symmetries. This particular model has a particle-hole symmetry which maps the Hamiltonian to minus itself. Particle-hole transformations must be antiunitary:

$$\mathcal{C} = \sigma^z \hat{K} \quad (16)$$

with $\hat{K} i \hat{K} = -i$. We have:

$$\mathcal{C}\hat{\mathcal{H}}\mathcal{C} = -\hat{\mathcal{H}} \quad (17)$$

So if we are expecting one state $|\psi\rangle$ with energy ϵ, we would also have another state:

$$|\psi'\rangle = \mathcal{C}|\psi\rangle, \, with \, \epsilon' = -\epsilon \quad (18)$$

But if there is only one state, then we must have $\epsilon = \epsilon' = 0$.

Armed with this we can look for the state:

$$E|\psi(x)\rangle = 0 = v\frac{1}{i}\sigma^z\frac{\partial|\psi(x)\rangle}{\partial x} + |g| sign(x)\sigma^x|\psi(x)\rangle \quad (19)$$

Let's guess:

$$|\psi(x)\rangle = \begin{pmatrix} u \\ v \end{pmatrix} e^{-\gamma|x|} \quad (20)$$

Why the u, v independent of the sign of x? Because the wave function must be cntinuous at $x = 0$. Plugging in we get:

$$(iv\gamma sign(x)\sigma^z + |g|sign(x)\sigma^x)\begin{pmatrix} u \\ v \end{pmatrix} = 0 \tag{21}$$

Multiplying from the left by the bracketed operator, we get:

$$(-\gamma^2 v^2 + g^2)\begin{pmatrix} u \\ v \end{pmatrix} = 0 \tag{22}$$

so $\gamma = g/v$ is obligatory. Putting this insight back in Eq. (21) we get:

$$|g|sign(x)\,(i\sigma^z + \sigma^x)\begin{pmatrix} u \\ v \end{pmatrix} = 0 \tag{23}$$

does this have a solution? Yes!

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} i \\ 1 \end{pmatrix} \tag{24}$$

So the mode is:

$$|\psi(x)\rangle = \frac{g/2v}{\sqrt{2}}\begin{pmatrix} i \\ 1 \end{pmatrix} e^{-|gx|/v} \tag{25}$$

with normalization even!

Is it symmetry under $\mathcal{C}$? Sure is! This is the Jackiw Rebbi protected mode. You can see that if you have a juncion with a $\Delta\phi \neq \pi$ the state loses its protection, and the hamiltonian loses the particle-hole symmetry. This shifts gradually the state from $\epsilon = 0$ towards the valence or conduction band, until when $\phi = 0$ it gets absorbed. This, along with time-reversal and chiral symmetries will be explored in the problem set.

C. 1d topological phase

We got a lot of mileage from the Thouless pump. But it deosnt' really make for a phase. we really suppressed any info on the exact model which is not expanded around the near-degeneracy point. It is time to overcome this issue to produce a true 1d topological phase - the Kitaev model. Kind of.

Consider a chain of atoms, with each having a single orbital for spinful electrons. The nearest neighbor hopping has an opposite sign for spin up and spin-down. Super strong spin-orbit coupling. Next, assume that there is an imaginary hopping between nn sites, which flip the spin. Superstrong Dresselhaus! What does the hamiltonian look like?

$$\begin{aligned} \hat{\mathcal{H}}\psi_n^\uparrow &= -J(\psi_{n-1}^\uparrow + \psi_{n+1}^\uparrow) - gi(\psi_{n-1}^\downarrow - \psi_{n+1}^\downarrow) \\ \hat{\mathcal{H}}\psi_n^\downarrow &= J(\psi_{n-1}^\downarrow + \psi_{n+1}^\downarrow) - gi(\psi_{n-1}^\uparrow - \psi_{n+1}^\uparrow) \end{aligned} \tag{26}$$

Fourier transform and write in terms of pauli:

$$\hat{\mathcal{H}} = -J\sigma^z \cos k + g \sin k \sigma^x \tag{27}$$

You can almost smell the topology on this one! The spectrum always contains a gap:

$$E_\pm(k) = \pm(J^2\cos^2 k + g^2 \sin^2 k)^{1/2} \tag{28}$$

This is like the distance from the origin of an ellipse [DRAW]

What is topolgoical? The spin goes all around. Can we make this model non-topological? Well, for that we need to add a parameter that can maket the ellipse not contain the origin. For that purpose we add what correspinds to a chemical potential in the superconducting problem that the kitaev model describe:

$$\hat{\mathcal{H}} = (-J\cos k - \mu)\sigma^z + g \sin k \sigma^x \tag{29}$$

If $|\mu| > J$, the ellipse endcoded in this hamiltoniain (again, treating the the pauli matrices as unit vectors) does not contain the origin any more.

What is the relation fo Jackiw and Rebbi to all these? Simple: the same physics arises at a phase boundary between different topological phases. So in the case of the Kitaev model, a domain wall between a topological phase with $|\mu| < J$ and a trivial one with $|\mu| > J$ is the existence of a protected sero energy state at the interace.

Furthermore, vacuum is considered a trivial phase. So if the system supproting a topological phase terminates somewhere, it will also have a protected zero energy state. For the Kitaev model, that's a Majorana state.

D. Edge mode of the Kitaev model

Let's assume that the chain only stretches between $x = 0$ and $x \to \infty$. In the topological phase there should be a Jackiw-Rebbi state at the $x = 0$ edge. Can we guess that it is going to be at zero? Let's tryto find a particle hole symmetry. Both $\mathcal{R} = \sigma^y$ and $\mathcal{C} = \hat{K}\sigma^x$ would give:

$$\mathcal{R}\hat{\mathcal{H}}\mathcal{R} = \mathcal{C}\hat{\mathcal{H}}\mathcal{C} = -\hat{\mathcal{H}} \tag{30}$$

So a single edge state would be pinned to $E = 0$. Let's find it.

If we guess an exponential decay, we can replace e^{ikn} with $e^{-\kappa n} = \zeta^n$. The solution, we guess, will have the form:

$$|\psi(x)\rangle = \zeta^n \begin{pmatrix} u \\ v \end{pmatrix} \tag{31}$$

The lattice SE (Eq. 26) then becomes:

$$\left[(-J(\zeta + 1/\zeta)/2 - \mu)\sigma^z - i\frac{g}{2}(\zeta - 1/\zeta)\sigma^x\right] \begin{pmatrix} u \\ v \end{pmatrix} = 0 \tag{32}$$

Which can only have a solution by:

$$-\frac{J}{2}(\zeta + 1/\zeta) - \mu = \pm\frac{g}{2}(\zeta - 1/\zeta) \to \zeta^2(J \pm g) + \mu\zeta + (J \mp g) = 0 \tag{33}$$

This would give a SE of the form:

$$(\sigma^z - \mp i\sigma^x) \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 1 & - \mp i \\ \pm i & -1 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = 0 \tag{34}$$

Which does have a solution:

$$(u,\, v) = (1,\, \mp i) \tag{35}$$

For ζ this yields

$$\zeta = \frac{\mu \pm \sqrt{\mu^2 - (J^2 - g^2)}}{J \pm g} \tag{36}$$

four solutions. Two of the solution actually belong to decaying solutions of the bound states on the right edge, with $|\zeta| > 1$. The other two must be $|\zeta| < 1$ with the same spinor.

But why two solutions? We need to satisfy boundary conditions. In first order difference equation, it is okay to require that the wave function vanishes at site $n = 0$ (the first site to the left of where the chain terminates). With a single exponent this is impossible. But if we have two solutions for ζ with the same spinor associated with them, then we can write the solution as:

$$|\psi(n)\rangle = (\zeta_1^n - \zeta_2^n) \begin{pmatrix} u \\ v \end{pmatrix}. \tag{37}$$

Choosing then the + option for the denominator of (36), we have the two solutions:

$$\zeta_{1,2} = \frac{\mu \pm \sqrt{\mu^2 - (J^2 - g^2)}}{J + g} \tag{38}$$

and $(u, v) = (1, -i)$.

Crucially, this is a σ^y eigenvalue. This is going to be very important below.

When do we stop having a zolution? When ζ touches 1. Indeed, substitute $\mu = J$ and you find it:

$$\zeta_{1,2} = \frac{J \pm g}{J + g} \tag{39}$$

and the edge state penetrate the bulk.