Ph135 - Problem Set 1

October 12, 2019

Problem 1

(a)

$$U = \int_0^\infty \rho_0 \epsilon \frac{1}{e^{\beta(\epsilon-\mu)}+1} d\epsilon \tag{1}$$

(b)

$$C_V = \left(\frac{\partial U}{\partial T}\right)_V, \tag{2}$$

Now, by the Sommerfeld expansion,

$$U = U_0 + \frac{\pi^2}{6} k_B^2 T^2 \rho_0. \tag{3}$$

This is expected, because the only place that electrons move around is T about the Fermi surface, and when they move around, they are excited by an amount T, so we have $U \propto T^2$. Therefore,

$$C_V = \frac{\pi^2}{3} \rho_0 k_B T \tag{4}$$

Problem 2

(a) Here, the electronic energies are:

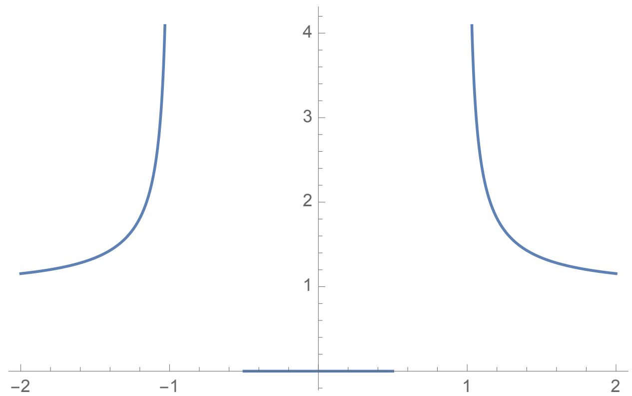
$$\tilde{\epsilon}_{\vec{k}} = \begin{cases} E_F + \sqrt{\Delta^2 + (\epsilon_{\vec{k}} - E_F)^2} & \epsilon_{\vec{k}} > E_F \\ E_F - \sqrt{\Delta^2 + (\epsilon_{\vec{k}} - E_F)^2} & \epsilon_{\vec{k}} < E_F \end{cases} \tag{5}$$

For small Δ,

$$\tilde{\epsilon}_{\vec{k}} = \begin{cases} E_F + |\epsilon_{\vec{k}} - E_F| \left(1 + \frac{\Delta^2}{(\epsilon_{\vec{k}} - E_F)^2}\right) & \epsilon_{\vec{k}} > E_F \\ E_F - |\epsilon_{\vec{k}} - E_F| \left(1 + \frac{\Delta^2}{(\epsilon_{\vec{k}} - E_F)^2}\right) & \epsilon_{\vec{k}} < E_F \end{cases} \tag{6}$$

and in the limit $\Delta \to 0$

$$\tilde{\epsilon}_{\vec{k}} = \begin{cases} \epsilon_{\vec{k}} & \epsilon_{\vec{k}} > E_F \\ \epsilon_{\vec{k}} & \epsilon_{\vec{k}} < E_F \end{cases} \tag{7}$$



(b)

$$\rho(E)dE = \rho(\epsilon_k)d\epsilon_k, \tag{8}$$

which implies that

$$\rho(E) = \rho_0 \left(\frac{dE}{d\epsilon_k}\right)^{-1}. \tag{9}$$

We plug in the expression in Eq 5, and get that

$$\rho(E) = \rho_0 \frac{|E|}{\sqrt{E^2 - \Delta^2}} \tag{10}$$

when $|E| \geq \Delta$. On the other hand, for $|E| < \Delta$, we get $\rho(E) = 0$ because $|E|$ cannot be smaller than Δ.

(c) The plot of density of states looks similar to Fig. 1.

(d)

$$U = \int dE\rho(E)\frac{E}{e^{\beta(E-\mu)}+1} = \int_{-\infty}^{-\Delta} dE\rho_0 \frac{-E^2}{\sqrt{E^2-\Delta^2}}\frac{1}{e^{\beta(E-\mu)}+1} \tag{11}$$

$$+ \int_{\Delta}^{\infty} dE\rho_0 \frac{E^2}{\sqrt{E^2-\Delta^2}}\frac{1}{e^{\beta(E-\mu)}+1} \tag{12}$$

(e) Here, $|E|$ is the excitation energy of BQPs, so $\mu = 0$ and thus:

$$U = \int dE\rho(E)\frac{E}{e^{\beta(E)}+1} = \int_{-\infty}^{-\Delta} dE\rho_0 \frac{-E^2}{\sqrt{E^2-\Delta^2}}\frac{1}{e^{\beta E}+1} \tag{13}$$

$$+ \int_{\Delta}^{\infty} dE\rho_0 \frac{E^2}{\sqrt{E^2-\Delta^2}}\frac{1}{e^{\beta E}+1} \tag{14}$$

$$U = \int_{-\infty}^{-\Delta} dE\rho_0 \frac{-E^2}{\sqrt{E^2-\Delta^2}}\left(1 + \frac{1}{e^{\beta E}+1} - 1\right) \tag{15}$$

$$+ \int_{\Delta}^{\infty} dE\rho_0 \frac{E^2}{\sqrt{E^2-\Delta^2}}\frac{1}{e^{\beta E}+1} \tag{16}$$

$$U = U(T=0) + \int_{-\infty}^{-\Delta} dE \rho_0 \frac{E^2}{\sqrt{E^2-\Delta^2}} \frac{1}{e^{-\beta E}+1} \tag{17}$$

$$+ \int_{\Delta}^{\infty} dE \rho_0 \frac{E^2}{\sqrt{E^2-\Delta^2}} \frac{1}{e^{\beta E}+1}, \tag{18}$$

which becomes

$$U = U(T=0) + 2\int_{\Delta}^{\infty} dE \rho_0 \frac{E^2}{\sqrt{E^2-\Delta^2}} \frac{1}{e^{\beta E}+1}. \tag{19}$$

Now, we make a change of variable $x \equiv \sqrt{E^2-\Delta^2}$, then

$$U = U(T=0) + \int_0^{\infty} dx \rho_0 \sqrt{x^2+\Delta^2} \frac{1}{e^{\beta(\sqrt{x^2+\Delta^2})}+1}. \tag{20}$$

When T is much less than Δ, we can truncate this integral at very small x and we can approximate $e^{\beta(\sqrt{x^2+\Delta^2})}+1 \approx e^{\beta(\sqrt{x^2+\Delta^2})}$, and $\sqrt{x^2+\Delta^2} \approx \Delta(1+\frac{x^2}{2\Delta^2})$ which gives

$$U \approx U(T=0) + \int_0^{\epsilon} dx \rho_0 (\Delta + \frac{x^2}{2\Delta}) e^{-\beta(\Delta + x^2/(2\Delta))} \tag{21}$$

$$\sim U(T=0) + \Delta^{3/2} T^{1/2} e^{-\Delta/T}, \tag{22}$$

and

$$C_V = (\frac{\partial U}{\partial T})_V \sim \Delta^{5/2} T^{-3/2} e^{-\Delta/T}. \tag{23}$$

We can see that because the lowest energy excitation has energy Δ, the heat capacity is suppressed exponentially with the Boltzmann factor.

Problem 3

At $T=0$, the total energy is given by

$$U = 2 \times \frac{1}{8} \times \int_0^{k_F} 4\pi k^2 \frac{V dk}{(2\pi)^3} \frac{\hbar^2 k^2}{2m}$$

$$= \frac{\hbar^2 k_F^5 V}{80\pi^2 m}. \tag{24}$$

Now

$$N = 2 \times \frac{1}{8} \times \frac{4}{3} \times \frac{k_F^3 V}{(2\pi)^3} = \frac{k_F^3 V}{24\pi^3}, \tag{25}$$

Therefore,

$$U = \frac{3}{5} N \frac{\hbar^2 k_F^2}{2m} = \frac{3}{5} N \frac{\hbar^2}{2m} \left(\frac{3\pi^2 N}{V}\right)^{2/3}, \tag{26}$$

and

$$P = -\frac{\partial U}{\partial V} = \frac{(3\pi^2)^{2/3}\hbar^2}{5m} \left(\frac{N}{V}\right)^{5/3} = \frac{(3\pi^2)^{2/3}\hbar^2}{5m} n^{5/3}. \tag{27}$$

For example, for copper, $n \approx 8.5 \times 10^{28}/m^3$, therefore,

$$P \approx 1.91 \times (1.05 \times 10^{-34})^2/(9 \times 10^{-31}) \times (8.5 \times 10^{28})^{5/3} \text{ Pa} \approx 3.85 \times 10^{10} \text{ Pa}. \tag{28}$$