# Ph135 - Problem Set 1

October 21, 2018

## Problem 1

(a)

$$U = \int_0^\infty \rho_0 \epsilon \frac{1}{e^{\beta(\epsilon-\mu)} + 1} d\epsilon \tag{1}$$

(b)

$$C_V = \left(\frac{\partial U}{\partial T}\right)_V, \tag{2}$$

Now by the Sommerfeld expansion,

$$U = U_0 + \frac{\pi^2}{6} k_B^2 T^2 \rho_0. \tag{3}$$

This is expected, because the only place that electrons move around is $T$ about the Fermi surface, and when they move around, they are excited by an amount $T$, so we have $U \propto T^2$. Therefore,

$$C_V = \frac{\pi^2}{3} \rho_0 k_B T \tag{4}$$

## Problem 2

(a)

$$\rho(E)dE = \rho(\epsilon_k)d\epsilon_k, \tag{5}$$

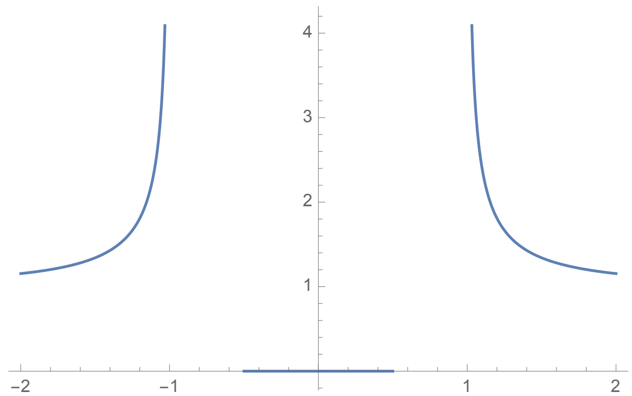
which implies that

$$\rho(E) = \rho_0 \left(\frac{dE}{d\epsilon_k}\right)^{-1}. \tag{6}$$

We plug in the expression in (1), and get that

$$\rho(E) = \rho_0 \frac{|E|}{\sqrt{E^2 - \Delta^2}} \tag{7}$$

when $|E| \geq \Delta$. $\rho(E) = 0$ when $|E| < \Delta$, because $|E|$ cannot be smaller than $\Delta$.




(b) The plot of density of states looks similar to Fig. 1.

(c)

$$U = \int dE \rho(E) \frac{E}{e^{\beta(E-\mu)}+1} = \int_{-\infty}^{-\Delta} dE \rho_0(E) \frac{-E^2}{\sqrt{E^2-\Delta^2}} \frac{1}{e^{\beta(E-\mu)}+1} \tag{8}$$

$$+ \int_{\Delta}^{\infty} dE \rho_0(E) \frac{E^2}{\sqrt{E^2-\Delta^2}} \frac{1}{e^{\beta(E-\mu)}+1} \tag{9}$$

(d) We do a change of variable $x \equiv \sqrt{E^2 - \Delta^2}$, then

$$U = \int_0^{\infty} dx \rho_0(x) \sqrt{x^2+\Delta^2} \frac{1}{e^{\beta(\sqrt{x^2+\Delta^2})}+1}. \tag{10}$$

When $T$ is much less than $\Delta$, we have $\epsilon_{\vec{k}} \sim E_F$, we can expand $U$ for small $x$, we get

$$\begin{aligned} U &\approx \int_0^{\epsilon} dx \rho_0(x)(\Delta + \frac{x^2}{2\Delta}) e^{-\beta(\Delta + x^2/(2\Delta))} \\ &\sim \Delta^{3/2} T^{1/2} e^{-\Delta/T}, \end{aligned} \tag{11}$$

and

$$C_V = (\frac{\partial U}{\partial T})_V \sim \Delta^{5/2} T^{-3/2} e^{-\Delta/T}. \tag{12}$$

We can see that because the lowest energy excitation has energy $\Delta$ the heat capacity is suppressed exponentially with the Boltzmann factor.

## Problem 3

At $T = 0$, the total energy is given by

$$\begin{aligned} U &= 2 \times \frac{1}{8} \times \int_0^{k_F} 4\pi k^2 \frac{V dk}{(2\pi)^3} \frac{\hbar^2 k^2}{2m} \\ &= \frac{\hbar^2 k_F^5 V}{80\pi^2 m}. \end{aligned} \tag{13}$$

Now

$$N = 2 \times \frac{1}{8} \times \frac{4}{3} \times \frac{k_F^3 V}{(2\pi)^3} = \frac{k_F^3 V}{24\pi^3}, \tag{14}$$

Therefore,

$$U = \frac{3}{5} N \frac{\hbar^2 k_F^2}{2m} = \frac{3}{5} N \frac{\hbar^2}{2m} \left( \frac{3\pi^2 N}{V} \right)^{2/3}, \tag{15}$$

and

$$P = -\frac{\partial U}{\partial V} = \frac{(3\pi^2)^{2/3} \hbar^2}{5m} \left( \frac{N}{V} \right)^{5/3} = \frac{(3\pi^2)^{2/3} \hbar^2}{5m} n^{5/3}. \tag{16}$$

For example, for copper, $n \approx 8.5 \times 10^{28}/m^3$, therefore,

$$P \approx 1.91 \times (1.05 \times 10^{-34})^2 / (9 \times 10^{-31}) \times (8.5 \times 10^{28})^{5/3} \text{ Pa} \approx 3.85 \times 10^{10} \text{ Pa}. \tag{17}$$