Ph135 - Problem Set 2

October 19, 2019

Problem 1

(a) We take the values of the magnetic field B at the minimum of the oscillations, 33.5T, 36.0T, 39.0T, 42.5T, 45.5T, 50.5T, 55.5T. We can estimate $\Delta(1/B)$ to be 0.002T^{-1}. Since

$$\Delta(\frac{1}{B}) = \frac{(2\pi)^2}{K\Phi_0} \tag{1}$$

The area in momentum space that a pocket contains is then

$$K = \frac{(2\pi)^2}{\Delta(1/B)\Phi_0} \approx \frac{39.44}{0.002 \times 2.07 \times 10^{-15}} m^{-2} \approx 9.53 \times 10^{18} m^{-2}. \tag{2}$$

(b) The electron density is given by

$$\begin{aligned} n &= 4 \times (2s+1)\frac{K}{(2\pi)^2} \\ &= 4 \times \frac{2K}{(2\pi)^2} \\ &\approx 1.93 \times 10^{18} m^{-2}. \end{aligned} \tag{3}$$

Problem2

1. Each state is occupied according the the Fermi-Dirac distribution. The energy density (energy per unit area in 2D) is:

$$u_B = 2\sum_{n=0}^{\infty} g\epsilon_n \frac{1}{e^{\beta(\epsilon_n-\mu)}+1} = 2\frac{B}{h/e}\sum_{n=0}^{\infty}\frac{\hbar\omega_c(n+1/2)}{e^{\beta(\hbar\omega_c(n+1/2)-\mu)}+1} \tag{4}$$

   2 for spin.

2. Consider the integral over the small range $[n\eta, (n+1)\eta]$:

$$\int_{n\eta}^{(n+1)\eta} f(x)dx \tag{5}$$

   expand the integrand around the midpoint $x_n = (n+\frac{1}{2})\eta$:

$$\int_{n\eta}^{(n+1)\eta} f(x)dx = \int_{n\eta}^{(n+1)\eta} f(x_n)dx + \int_{n\eta}^{(n+1)\eta} f'(x_n)\cdot(x-x_n)dx + \int_{n\eta}^{(n+1)\eta} f''(x_n)\frac{(x-x_n)^2}{2}dx + \dots \tag{6}$$

now do the integration:

$$
\begin{aligned}
&= \eta f(x_n) + 0 + \frac{1}{2}\left.\frac{(x-x_n)^3}{3}\right|_{n\eta}^{(n+1)\eta} + \dots \\
&= \eta f((n+\frac{1}{2})\eta) + 0 + \frac{\eta^2}{24}\eta f''((n+\frac{1}{2})\eta) + \dots \qquad (7)
\end{aligned}
$$

because of the explicit η^2 in the last term, we can write it as:

$$
\frac{\eta^2}{24}\eta f''((n+\frac{1}{2})\eta) = \frac{\eta^2}{24}\int_{n\eta}^{(n+1)\eta} f''(x)dx \qquad (8)
$$

where we mean the two sides are equal to lowest order in η which is η^3. Now summing over n gives:

$$
\eta\sum_{n=0}^{\infty} f(\eta(\frac{1}{2}+n)) \approx \int_0^{\infty} dx f(x) - \frac{\eta^2}{24}\int_0^{\infty} f''(x)dx + \mathcal{O}(\eta^4) \qquad (9)
$$

3. We are after the difference between of energies between electrons in a magnetic field and free electrons at constant temperature and chemical potential.
The energy density of the free electron gas is:

$$
\begin{aligned}
u_{free} &= \int_0^{\infty} \rho(\epsilon)\epsilon\frac{1}{e^{\beta(\epsilon-\mu)}+1}d\epsilon = \frac{4\pi m}{h^2}\int_0^{\infty}\epsilon\frac{1}{e^{\beta(\epsilon-\mu)}+1}d\epsilon \\
&= \frac{4\pi m}{h^2}\frac{1}{\beta^2}\int_0^{\infty} x\frac{1}{e^{(x-\beta\mu)}+1}dx \qquad (10)
\end{aligned}
$$

the density of states $\rho(\epsilon)$ in 2D is constant and equal to $4\pi m/h^2$ (including spin, see lecture notes). It is worth noticing that for 2D case, $\rho(\epsilon)\hbar\omega_c = 2\frac{B}{h/e}$ which shows that all the states in an energy interval of $\hbar\omega_c$ collapse into one Landau level which has a degeneracy of $2\frac{B}{h/e}$(including the spin degeneracy).

To approximate the energy in magnetic field, we need to get the expression from part (a) to the form of part (b). We can do this by identifying $\hbar\omega\beta$ as η:

$$
\begin{aligned}
u_B &= 2\frac{B}{h/e}\sum_{n=0}^{\infty}\frac{\hbar\omega_c(n+1/2)}{e^{\beta(\hbar\omega_c(n+1/2)-\mu)}+1} = \left(2\frac{B}{h/e}\frac{1}{\hbar\omega_c\beta^2}\right)\hbar\omega_c\beta\sum_{n=0}^{\infty}\frac{\hbar\omega_c(n+1/2)\beta}{e^{\beta(\hbar\omega_c(n+1/2)-\mu)}+1} \qquad (11) \\
&= \frac{4\pi m}{h^2}\frac{1}{\beta^2}\eta\sum_{n=0}^{\infty} f(\eta(n+1/2)) \qquad (12)
\end{aligned}
$$

n the second line we put in $\omega_c = eB/m$ and identified:

$$
\begin{aligned}
\eta &= \hbar\omega_c\beta \qquad (13) \\
f(x) &= \frac{x}{e^{x-\beta\mu}+1} \qquad (14)
\end{aligned}
$$

Now we are in a position to approximate this expression using part (b):

$$
u_B \approx \frac{4\pi m}{h^2}\frac{1}{\beta^2}\left[\int_0^{\infty} x\frac{1}{e^{(x-\beta\mu)}+1}dx - \frac{\eta^2}{24}\int_0^{\infty} f''(x)dx\right] \qquad (15)
$$

The first term is exactly the energy of the free electron gas, so taking the difference leaves the second term only:

$$
\begin{aligned}
u_B - u_{free} &= -\frac{4\pi m}{h^2}\frac{1}{\beta^2}\frac{\eta^2}{24}\int_0^\infty f''(x)dx = -\frac{4\pi m}{h^2}\frac{1}{\beta^2}\frac{(\hbar\omega_c\beta)^2}{24}\ f'(x)\Big|_0^\infty \\
&= \frac{e^2}{24\pi m}B^2\frac{1}{e^{-\beta\mu}+1} \Rightarrow_{T\to 0} \frac{e^2}{24\pi m}B^2 \qquad (16)
\end{aligned}
$$

The Landau magnetic susceptibility is therefore:

$$
\chi_L = -\frac{e^2}{24\pi m} \qquad (17)
$$