Ph135 - Problem Set 6

November 28, 2019

Problem1

(a)

$$H = \frac{p^2}{2m} + \beta(\sigma^x p_x - \sigma^y p_y) + \alpha(\sigma^x p_y - \sigma^y p_x), \tag{1}$$

so

$$\begin{aligned}(H - \frac{p^2}{2m})^2 &= \Big[\beta(\sigma^x p_x - \sigma^y p_y) + \alpha(\sigma^x p_y - \sigma^y p_x)\Big]^2 \\ &= (\alpha^2 + \beta^2)p^2 + \alpha\beta(\sigma^x p_x - \sigma^y p_y)(\sigma^x p_y - \sigma^y p_x) + \alpha\beta(\sigma^x p_y - \sigma^y p_x)(\sigma^x p_x - \sigma^y p_y) \\ &= (\alpha^2 + \beta^2)p^2 + 4\alpha\beta p_x p_y. \end{aligned} \tag{2}$$

Therefore,

$$E(p) = \frac{p^2}{2m} \pm \sqrt{(\alpha^2 + \beta^2)p^2 + 4\alpha\beta p_x p_y}. \tag{3}$$

(b) The two bands cross when

$$(\alpha^2 + \beta^2)p^2 + 4\alpha\beta p_x p_y = 0, \tag{4}$$

which is equivalent to

$$\begin{aligned}&(\alpha p_x + \beta p_y)^2 + (\alpha p_y + \beta p_x)^2 \\ &= \begin{pmatrix} \alpha & \beta \\ \beta & \alpha \end{pmatrix} \begin{pmatrix} p_x \\ p_y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \end{aligned} \tag{5}$$

The equation has nontrivial solution if and only if the determinant of the matrix $\alpha^2 - \beta^2 = 0$. Therefore, the electronic energy states split into two non-degenerate bands for all α and β except when $\alpha = \pm\beta$.

(c) We consider three limits:

(1) $\beta \gg \alpha$, in which case

$$H \approx \frac{p^2}{2m} + \beta(\sigma^x p_x - \sigma^y p_y), \tag{6}$$

and at momentum (p_x, p_y), the spin points along $(p_x, -p_y)$, so the electron spin makes 1 turn around $p = 0$ in the clockwise direction, and the winding number is -1.

(2) $\beta \ll \alpha$, in which case

$$H \approx \frac{p^2}{2m} + \alpha(\sigma^x p_y - \sigma^y p_x), \tag{7}$$

and at momentum (p_x, p_y), the spin points along $(-p_x, p_y)$, so the electron spin makes 1 turn around $p = 0$ in the counterclockwise direction, and the winding number is 1.

(c) $\beta = \alpha$, in which case

$$H = \frac{p^2}{2m} + \alpha(\sigma^x p_x - \sigma^y p_y) + \alpha(\sigma^x p_y - \sigma^y p_x), \tag{8}$$

and at momentum (p_x, p_y), the spin points along $(p_x + p_y, -p_x - p_y)$, so the electron spin makes 0 turn around $p = 0$, and the winding number is 0.

Since the winding number is an integer, we can interpolate between limits and get that the winding number of the spin is given by

$$n = \begin{cases} 1, & \alpha > \beta, \\ 0, & \alpha = \beta, \\ -1, & \alpha < \beta. \end{cases} \tag{9}$$

Problem 3

(a) A spin 1 undergoes a rotation such that

$$|\Psi\,\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 + \cos t \\ \sqrt{2}\sin t \\ 1 - \cos t \end{pmatrix}. \tag{10}$$

Now, the Berry phase acquired during the adiabatic evolution from $t = 0$ to $t = 2\pi$ is given by

$$\gamma = -i \int_{t=0}^{t=2\pi} dt \langle \Psi\,|\partial_t|\Psi\,\rangle = 0. \tag{11}$$

It is worth noticing that the above formula for Berry phase is valid only if the given state $|\Psi\,\rangle$ is normalized for all values of t. If $< \Psi|\Psi >\neq 1$, then $\gamma = -\text{Img}(\int_{t=0}^{t=2\pi} dt \langle \Psi\,|\partial_t|\Psi\,\rangle)$.

(b) Here, the wavefunction is given by:

$$|\Psi\,\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} (1 + \cos t)e^{it} \\ \sqrt{2}\sin t \\ (1 - \cos t)e^{-it} \end{pmatrix}. \tag{12}$$

and the Berry Phase accumulated during the adiabatic time evolution from $t = 0$ to $t = \pi/2$ is

$$\gamma = -i \int_{t=0}^{t=\pi/2} dt \langle \Psi\,|\partial_t|\Psi\,\rangle = \int_{t=0}^{t=\pi/2} (2\cos t)dt = -2 \tag{13}$$