Ph135 - Problem Set 7

December 4, 2018

Problem 1

(a) The number of electron states is given by

$$\begin{aligned} N_D &= \int_{-\infty}^{\infty} \rho(\epsilon) \frac{1}{e^{\beta(\epsilon-\mu)}+1} d\epsilon \\ &= N_D \frac{1}{e^{\beta(-\delta-\mu)}+1} + \rho_0 \int_0^{\infty} \frac{1}{e^{\beta(\epsilon-\mu)}+1} d\epsilon \\ &= N_D \frac{1}{e^{\beta(-\delta-\mu)}+1} + \rho_0 T \ln(1+e^{\beta\mu}). \end{aligned} \tag{1}$$

Therefore,

$$N_D\left(1 - \frac{1}{e^{\beta(-\delta-\mu)}+1}\right) = \rho_0 T \ln(1+e^{\beta\mu}), \tag{2}$$

which implies that

$$\rho_0 T \ln(1+e^{\beta\mu}) = N_D \frac{1}{e^{\beta(\delta+\mu)}+1}. \tag{3}$$

(b) When $T \ll \delta$, $-\delta < \mu < 0$, so

$$N_D \frac{1}{e^{\beta(\delta+\mu)}+1} \approx N_D e^{-\beta(\delta+\mu)}, \tag{4}$$

and

$$\rho_0 T \ln(1+e^{\beta\mu}) \approx \rho_0 T e^{\beta\mu}. \tag{5}$$

(3) Therefore reduces to

$$\rho_0 T e^{2\beta\mu} \approx N_D e^{-\beta\delta}, \tag{6}$$

and

$$\mu \approx -\frac{\delta}{2} + \frac{T}{2} \ln \frac{N_D}{\rho_0 T}. \tag{7}$$

(c) When $T \gg \delta$,

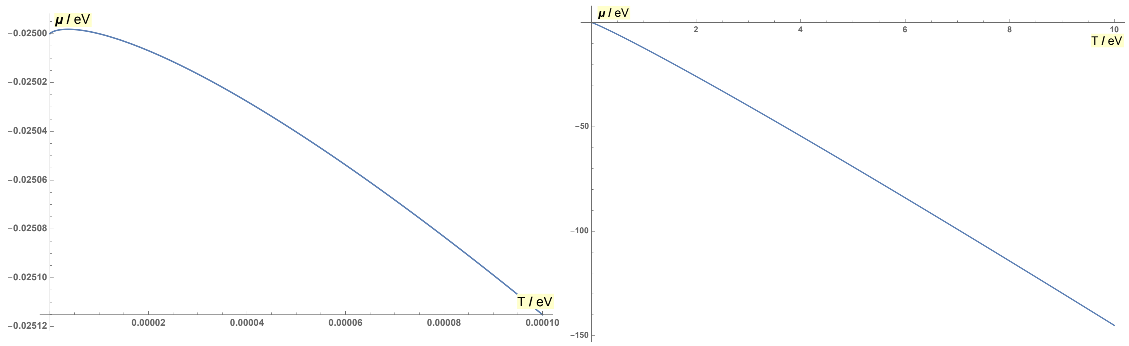
$$N_D \frac{1}{e^{\beta(\delta+\mu)}+1} \approx \frac{1}{2} N_D, \tag{8}$$

and so

$$\rho_0 T \ln(1+e^{\beta\mu}) \approx \frac{1}{2} N_D, \tag{9}$$

and

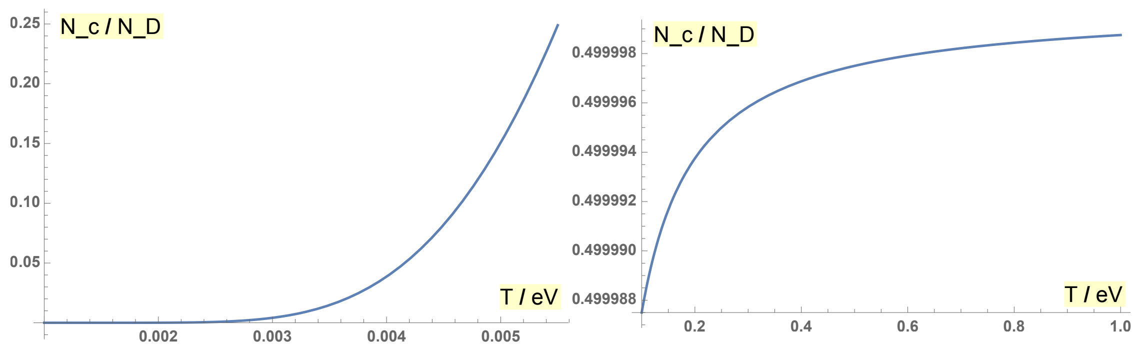
$$\begin{aligned} \mu &\approx T \ln(e^{N_D/(2\rho_0 T)} - 1) \\ &\approx T \ln\left(\frac{N_D}{2\rho_0 T}\right). \end{aligned} \tag{10}$$



(a) $T \ll \delta$

(b) $T \gg \delta$

Figure 1: Plot of the chemical potential as a function of T in the regime $T \ll \delta$ (left), and $T \gg \delta$ (right).



(a) $T \ll \delta$

(b) $T \gg \delta$

Figure 2: Plot of the number of carriers as a function of T in the regime $T \ll \delta$ (left), and $T \gg \delta$ (right).

(d) See Fig. 1 and Fig. 2.

(e) When $T \ll \delta$,

$$
\begin{aligned}
e^{\beta\mu} &\approx \exp\Big[-\frac{\beta\delta}{2} + \frac{1}{2}\ln\frac{N_D}{\rho_0 T}\Big] \\
&= \sqrt{\frac{N_D}{\rho_0 T}}e^{-\beta\delta/2}.
\end{aligned}
\tag{11}
$$

Therefore,

$$
\begin{aligned}
N_{\text{carriers}} &= \rho_0 T \ln(1 + e^{\beta\mu}) \\
&\approx \rho_0 T e^{\beta\mu} \\
&\approx \sqrt{N_D \rho_0 T} e^{-\delta/(2T)}.
\end{aligned}
\tag{12}
$$

When $T \gg \delta$,

$$
\begin{aligned}
e^{\beta\mu} &\approx e^{\beta T \ln(\frac{N_D}{2\rho_0 T})} \\
&= \frac{N_D}{2\rho_0 T}.
\end{aligned}
\tag{13}
$$

Therefore,

$$
\begin{aligned}
N_{\text{carriers}} &= \rho_0 T \ln(1 + e^{\beta\mu}) \\
&\approx \rho_0 T \ln(1 + \frac{N_D}{2\rho_0 T}) \\
&\approx \rho_0 T \left[\frac{N_D}{2\rho_0 T} - \frac{1}{2}\left(\frac{N_D}{2\rho_0 T}\right)^2 \right] \\
&\approx \frac{N_D}{2} - \frac{1}{8}\frac{N_D^2}{\rho_0 T}.
\end{aligned} \tag{14}
$$

Problem 2

(a)

$$
H = \frac{p^2}{2m} + \beta(\sigma^x p_x - \sigma^y p_y) + \alpha(\sigma^x p_y - \sigma^y p_x), \tag{15}
$$

so

$$
\begin{aligned}
(H - \frac{p^2}{2m})^2 &= \Big[\beta(\sigma^x p_x - \sigma^y p_y) + \alpha(\sigma^x p_y - \sigma^y p_x)\Big]^2 \\
&= (\alpha^2 + \beta^2)p^2 + \alpha\beta(\sigma^x p_x - \sigma^y p_y)(\sigma^x p_y - \sigma^y p_x) + \alpha\beta(\sigma^x p_y - \sigma^y p_x)(\sigma^x p_x - \sigma^y p_y) \\
&= (\alpha^2 + \beta^2)p^2 + 4\alpha\beta p_x p_y.
\end{aligned} \tag{16}
$$

Therefore,

$$
E(p) = \frac{p^2}{2m} \pm \sqrt{(\alpha^2 + \beta^2)p^2 + 4\alpha\beta p_x p_y}. \tag{17}
$$

(b) The two bands cross when

$$
(\alpha^2 + \beta^2)p^2 + 4\alpha\beta p_x p_y = 0, \tag{18}
$$

which is equivalent to

$$
\begin{aligned}
&(\alpha p_x + \beta p_y)^2 + (\alpha p_y + \beta p_x)^2 \\
&= \begin{pmatrix} \alpha & \beta \\ \beta & \alpha \end{pmatrix} \begin{pmatrix} p_x \\ p_y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.
\end{aligned} \tag{19}
$$

The equation has nontrivial solution if and only if the determinant of the matrix $\alpha^2 - \beta^2 = 0$. Therefore, the electronic energy states split into two non-degenerate bands for all α and β except when $\alpha = \pm\beta$.

(c) We consider three limits:

(1) $\beta \gg \alpha$, in which case

$$
H \approx \frac{p^2}{2m} + \beta(\sigma^x p_x - \sigma^y p_y), \tag{20}
$$

and at momentum (p_x, p_y), the spin points along $(p_x, -p_y)$, so the electron spin makes 1 turn around $p = 0$ in the clockwise direction, and the winding number is -1.

(2) $\beta \ll \alpha$, in which case

$$
H \approx \frac{p^2}{2m} + \alpha(\sigma^x p_y - \sigma^y p_x), \tag{21}
$$

and at momentum (p_x, p_y), the spin points along $(-p_x, p_y)$, so the electron spin makes 1 turn around $p = 0$ in the counterclockwise direction, and the winding number is 1.

(c) $\beta = \alpha$, in which case

$$H = \frac{p^2}{2m} + \alpha(\sigma^x p_x - \sigma^y p_y) + \alpha(\sigma^x p_y - \sigma^y p_x), \tag{22}$$

and at momentum (p_x, p_y), the spin points along $(p_x + p_y, -p_x - p_y)$, so the electron spin makes 0 turn around $p = 0$, and the winding number is 0.

Since the winding number is an integer, we can interpolate between limits and get that the winding number of the spin is given by

$$n = \begin{cases} 1, & \alpha > \beta, \\ 0, & \alpha = \beta, \\ -1, & \alpha < \beta. \end{cases} \tag{23}$$