Ph135 - Problem Set 7

December 9, 2019

Problem 1

The Hamiltonian is given by

$$\hat{H} = \sigma_z v\hat{p} + g\sigma_x \cos\frac{\gamma}{2} + g\cdot \mathrm{sgn}(x)\sigma_y \sin\frac{\gamma}{2}. \tag{1}$$

We find its localized mode, we use the ansatz

$$\Psi = \begin{pmatrix} u \\ w \end{pmatrix} e^{-|\kappa|x}. \tag{2}$$

Substitute into the equation above, we get

$$i\kappa v\cdot \mathrm{sgn}(x)\begin{pmatrix} u \\ -w \end{pmatrix} + g\cos\frac{\gamma}{2}\begin{pmatrix} w \\ u \end{pmatrix} + g\sin\frac{\gamma}{2}\cdot \mathrm{sgn}(x)\begin{pmatrix} -iw \\ iu \end{pmatrix} = E\begin{pmatrix} u \\ w \end{pmatrix}. \tag{3}$$

The imaginary part gives

$$\kappa v\begin{pmatrix} u \\ -w \end{pmatrix} + g\sin\frac{\gamma}{2}\begin{pmatrix} -w \\ u \end{pmatrix} = 0, \tag{4}$$

The nontrivial solutions are given by

$$\begin{aligned} &u = w, \quad \kappa = \frac{g}{v}\sin\frac{\gamma}{2}, \quad \text{if } \frac{g}{v}\sin\frac{\gamma}{2} \geq 0 \\ &u = -w, \quad \kappa = -\frac{g}{v}\sin\frac{\gamma}{2}, \quad \text{otherwise.} \end{aligned} \tag{5}$$

Plug the solutions into the real part of the equation, we find that

$$g\cos\frac{\gamma}{2}\begin{pmatrix} v \\ u \end{pmatrix} = E\begin{pmatrix} v \\ u \end{pmatrix}, \tag{6}$$

so the energy of the localized modes are given by

$$\begin{aligned} &E = g\cos\frac{\gamma}{2}, \quad \text{if } \frac{g}{v}\sin\frac{\gamma}{2} \geq 0 \\ &E = -g\cos\frac{\gamma}{2}, \quad \text{otherwise.} \end{aligned} \tag{7}$$

To summarize, when $g\sin(\gamma/2)/v \geq 0$, the localized mode is given by

$$\Psi(x) = \frac{v}{\sqrt{2}g\sin(\gamma/2)} e^{-g\sin(\gamma/2)|x|/v}\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \tag{8}$$

with energy

$$E = g\cos\frac{\gamma}{2}. \tag{9}$$

When $g\sin(\gamma/2)/v < 0$, the localized mode is given by

$$\Psi(x) = \frac{v}{\sqrt{2}g\sin(\gamma/2)} e^{g\sin(\gamma/2)|x|/v} \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \tag{10}$$

with energy

$$E = -g\cos\frac{\gamma}{2}. \tag{11}$$

Problem 2

For $H = \sigma_z v\hat{p}$, the right-moving electrons have $E_R = vp$, and the left-moving electrons have $E_L = -vp$. The band structure is as illustrated in Figure 1. The number of uncompensated states is therefore given by

$$\begin{aligned} N &= \frac{L}{2\pi}\int_{-eV/(2\hbar v)}^{eV/(2\hbar v)} dk \\ &= \frac{eVL}{2\pi\hbar v} = \frac{eVL}{hv}. \end{aligned} \tag{12}$$

The charge density is

$$e\frac{N}{L} = \frac{e^2V}{hv}. \tag{13}$$

Therefore the current is given by the velocity times the density, which is

$$j = \frac{e^2V}{h}, \tag{14}$$

and the conductance is

$$\sigma = \frac{j}{V} = \frac{e^2}{h}. \tag{15}$$

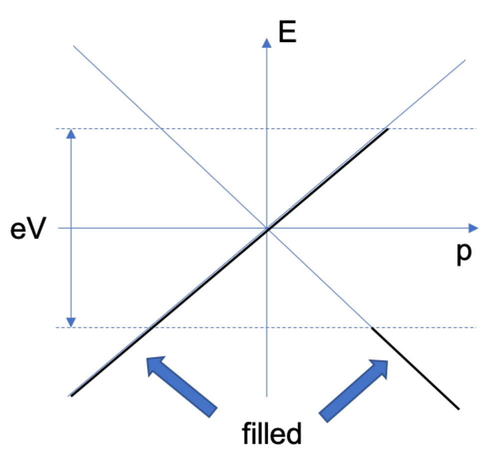

Figure 1: Illustration of the band structure