Ph135 - Problem Set 7

December 9, 2019

Problem 1

The Hamiltonian is given by

\[ \hat{H} = \sigma_z v \hat{p} + g \sigma_x \cos \frac{\gamma}{2} + g \cdot \text{sgn}(x) \sigma_y \sin \frac{\gamma}{2}. \]  

(1)

We find its localized mode, we use the ansatz

\[ \Psi = \begin{pmatrix} u \\ w \end{pmatrix} e^{-|\kappa|x}. \]  

(2)

Substitute into the equation above, we get

\[ \begin{align*}
    i\kappa v \cdot \text{sgn}(x) 
        \left( \begin{array}{c} u \\
             -w 
        \end{array} \right) 
    + g \cos \frac{\gamma}{2} 
        \left( \begin{array}{c} w \\
             u 
        \end{array} \right) 
    + g \sin \frac{\gamma}{2} \cdot \text{sgn}(x) 
        \left( \begin{array}{c} -iw \\
             iu 
        \end{array} \right) 
    = E \begin{pmatrix} u \\ w \end{pmatrix}. 
\end{align*} \]  

(3)

The imaginary part gives

\[ \kappa v \left( \begin{array}{c} u \\
             -w 
        \end{array} \right) 
    + g \sin \frac{\gamma}{2} \left( \begin{array}{c} w \\
             -u 
        \end{array} \right) 
    = 0, \]  

(4)

The nontrivial solutions are given by

\[ \begin{align*}
    u &= w, \quad \kappa = \frac{g}{v} \sin \frac{\gamma}{2}, \quad \text{if } \frac{g}{v} \sin \frac{\gamma}{2} \geq 0 \\
    u &= -w, \quad \kappa = -\frac{g}{v} \sin \frac{\gamma}{2}, \quad \text{otherwise}. 
\end{align*} \]  

(5)

Plug the solutions into the real part of the equation, we find that

\[ g \cos \frac{\gamma}{2} \left( \begin{array}{c} v \\
             u 
        \end{array} \right) = E \begin{pmatrix} v \\ u \end{pmatrix}, \]  

(6)

so the energy of the localized modes are given by

\[ \begin{align*}
    E &= g \cos \frac{\gamma}{2}, \quad \text{if } \frac{g}{v} \sin \frac{\gamma}{2} \geq 0 \\
    E &= -g \cos \frac{\gamma}{2}, \quad \text{otherwise}. 
\end{align*} \]  

(7)

To summarize, when \( g \sin (\gamma/2)/v \geq 0 \), the localized mode is given by

\[ \Psi(x) = \frac{v}{\sqrt{2g \sin (\gamma/2)}} e^{-g \sin (\gamma/2)|x|/v} \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \]  

(8)

with energy

\[ E = g \cos \frac{\gamma}{2}. \]  

(9)
When \( g \sin (\gamma/2)/v < 0 \), the localized mode is given by

\[
\Psi(x) = \frac{v}{\sqrt{2g \sin (\gamma/2)}} e^{g \sin (\gamma/2)x/v} \begin{pmatrix} 1 \\ -1 \end{pmatrix},
\]

with energy

\[
E = -g \cos \frac{\gamma}{2}.
\]

**Problem 2**

For \( H = \sigma_z v \hat{p} \), the right-moving electrons have \( E_R = vp \), and the left-moving electrons have \( E_L = -vp \). The band structure is as illustrated in Figure 1. The number of uncompensated states is therefore given by

\[
N = \frac{L}{2\pi} \int_{-eV/(2\hbar v)}^{eV/(2\hbar v)} dk = \frac{eVL}{2\pi \hbar v} = \frac{eVL}{hV}.
\]

The charge density is

\[
e \frac{N}{L} = \frac{e^2 V}{hV}.
\]

Therefore the current is given by the velocity times the density, which is

\[
j = \frac{e^2 V}{h},
\]

and the conductance is

\[
\sigma = \frac{j}{V} = \frac{e^2}{h}.
\]

![Figure 1: Illustration of the band structure](Image)