Sommerfeld expansion

The zero temperature properties of the fermi-dirac allowed us to calculate the pressure, Fermi-energy, and the Pauli susceptibility, as well as the Shubnikov de-Haas oscillations, in principle, we could also calculate the Landau diamagnetism, but we left it for later. But how do things change with temperature? How does temperature come in? For this part of class we turn the magnetic field off, and concentrate on temperature effects.

Let's start with a simple quantity - the heat capacity. What would be its dependence on temperature? Before doing a hard-core calculation, let's use the intuition from the Fermi-Dirac ditribution to find it.

If we have a low temperature -

$$T \ll E_F \tag{1}$$

then the fermi-Dirac distribution is 1 or zero outside a region were:

$$e^{(\epsilon_{\vec{k}}-\mu)/T} \tag{2}$$

is not zero or infinity. This region is of thickness $2T$. This is where everything happens at finite temperature. So what is the temperature dependent energy?

At zero temperature we know what the energy is:

$$U = \int_0^{E_F} d\epsilon \rho(\epsilon)\epsilon \tag{3}$$

But this must have low T corrections, which we can presumably calculate as a power series in T. I want ot show you how to get the first corrections simply on the basis of geometry and intution.

We can think of the distribution as essentially a straight line going from zero at $E_F + T$ to 1 at $E_F - T$. So half of the states up here (DRAW THE triangles and the shifted mass) move to higher energy. How many states are we talking about?

$$\overline{n} \approx \rho(E_F)T/4 \tag{4}$$

The energy gain is twice the average energy in each triangle. The center of mass of a triangle is at a third of its height. here it means that the left triangle is $T/3$ away from E_F, and after it moves to the other side, it is displaced, on average the same amount to the right. Thus the energy contribution is:

$$\Delta U = \frac{2}{3}\overline{n}T = \frac{1}{6}\rho(E_F)T^2 \tag{5}$$

This is a T^2 correction to the energy of the degenerate fermi-gas. In fact, we can continue this way, and it turns out that the heat capacity will only have even power of T.

The heat capacity at constant volume is just:

$$\frac{dU}{dT} \sim \rho(E_F)T \tag{6}$$

This is one of the earmarks of an electron gas in metals, and of fermi-liquids in general.

We can't really trust the coefficient completely, but we get the correct temperture dependence.

Now that we got more of a feelign for the fermi dirac distribution, let's derive a systematic way to produce the temperature dependence of quantities relating to fermi gasses. This is the Sommerfeld approximation.

What are the interesting TD quantities for an electron gas?

The energy integral for instance, is:

$$E = V\int_0^{\infty} d\epsilon \rho(\epsilon)\frac{\epsilon}{e^{(\epsilon-\mu)/T}+1} \tag{7}$$

The number integral is:

$$N = V\int_0^{\infty} d\epsilon \rho(\epsilon)\frac{1}{e^{(\epsilon-\mu)/T}+1} \tag{8}$$

The general thing seems to be:

$$G/V = \int_0^\infty d\epsilon g(\epsilon) \frac{1}{e^{(\epsilon-\mu)/T}+1} \tag{9}$$

We would like to do to this integral analytically, what we just did to it graphically above, with the heat capacity. The idea is that we think of the FD distribution as a step, and then go back to reconsider:

$$G/V = \int_0^\mu d\epsilon g(\epsilon) + \int_0^\mu d\epsilon g(\epsilon) \left(\frac{1}{e^{(\epsilon-\mu)/T}+1} - 1\right) + \int_\mu^\infty d\epsilon \rho(\epsilon) \frac{1}{e^{(\epsilon-\mu)/T}+1} \tag{10}$$

So we separated the inegral into three parts - the integral upto the chemical potential assuming occupation 1. a correction for the energy parts that are from small energies, and correction from higher energies, suppressed by the distribution. We treat first the second part - carrying out the algebra in the brackets, it is just:

$$\left(\frac{1}{e^{(\epsilon-\mu)/T}+1} - 1\right) = -\frac{1}{e^{-(\epsilon-\mu)\beta}+1} \tag{11}$$

The other thing is - we change the integration limit such that instead of being integrated from zero to μ, we kinda cheat and allow integration from $-\infty$ to μ. This is inly going to hurt us exponentially if at all.

The next steps are: change the integration variables so that they are relative to μ: $\epsilon \to \mu + \epsilon$. This means:

$$G/V = G_0/V - \int_{-\infty}^0 d\epsilon g(\mu+\epsilon) \frac{1}{e^{-\epsilon/T}+1} + \int_0^\infty d\epsilon \rho(\mu+\epsilon) \frac{1}{e^{\epsilon/T}+1} \tag{12}$$

Next, we change the integration variable in the first integral to minus itself: $\epsilon \to -\epsilon$. You can see why - right? The expressions are just very similar on these two. This is the symmetric property of the FD distribution - take off 1, and then change right to left, and you got the same thing. We get:

$$\to G_0/V - \int_0^\infty d\epsilon g(\mu-\epsilon) \frac{1}{e^{\epsilon/T}+1} + \int_0^\infty d\epsilon \rho(\mu+\epsilon) \frac{1}{e^{\epsilon/T}+1} = G_0/V + \int_0^\infty d\epsilon \left(g(\mu+\epsilon) - g(\mu-\epsilon)\right) \frac{1}{e^{\epsilon/T}+1} \tag{13}$$

The next step is to expend the function $g(\epsilon)$ in a taylor series. Only odd powers of ϵ seem to survive:

$$g(\mu+\epsilon) - g(\mu-\epsilon) = 2(g'(\mu)\epsilon + \frac{1}{3!} g'''(\mu)\epsilon^3 \ldots) = 2 \sum_{n=1(odd)}^{\infty} \frac{1}{n!} g^{(n)}(\mu)\epsilon^n \tag{14}$$

To finish the expansion we consider a general term in the integral. Once we put the taylor expansion in we would get:

$$g^{(n)}(\mu) \int_0^\infty d\epsilon \epsilon^n \frac{1}{e^{\epsilon/T}+1} \tag{15}$$

Here, we really want the temperature dependence. So we rescale, $\epsilon = xT$:

$$\to g^{(n)}(\mu) T^{n+1} \int_0^\infty dx \frac{x^n}{e^x+1} \tag{16}$$

This is what I wanted - the integral now only contains numbers, and is just a numeber. we can get a series expansion for it pretty easily by expanding the denominator in the exponent:

$$f_n = \int_0^\infty dx \frac{x^n}{e^x+1} = \int_0^\infty dx x^n e^{-x}(1 - e^{-x} + e^{-2x} - e^{-3x} \ldots) = n! \left(1 - \frac{1}{2^{n+1}} + \frac{1}{3^{n+1}} \ldots\right) \tag{17}$$

But the general expression is now:

$$G_0/V + \sum_{n>0,odd} T^{n+1} \frac{2}{n!} g^{(n)}(\mu) f_n \tag{18}$$

very simple...

For the energy we get -

$$E/V = E_0/V + 2T^2 \left. \frac{d}{d\epsilon}(\epsilon\rho(\epsilon)) \right|_{\epsilon=\mu} f_1 = E_0/V + 2f_1 T^2 (\rho(\mu) + \mu\rho'(\mu)) \tag{19}$$

same T dependence as we found, but not quite the same number.

That's it for Fermions. Fascinating particles. You try to cool them down to absolute zero, and they move at speeds that makes your ferarri blush. you try to confine them into a block of iron, and they press against the walls of their confinement with pressure that would crush us to little crums. They dictate the behavior of cold metals, and that of extremely hot white dwarves. amazing.