Ph135 - Final Exam

December 17, 2018

Problem 1

(a) The potential has periodicity a, because $V(x+ma) = V(x)$, for any $m \in \mathbb{Z}$.

(b) The Fourier components are given by

$$\begin{aligned} V_m &= \frac{1}{a}\int_0^a V(x)e^{-imkx}dx \\ &= -\frac{1}{a}\int_0^a \sum_{n=-\infty}^{\infty} \frac{V}{(x-na)^2+\lambda^2} e^{-imkx}\, dx. \end{aligned} \tag{1}$$

Let $y = x/a$, then

$$\begin{aligned} V_m &= -\int_0^1 \sum_{n=-\infty}^{\infty} \frac{V}{(y-n)^2+\lambda^2/a^2} e^{-im2\pi y}\, dy \\ &= -\int_{-\infty}^{\infty} \frac{V}{y^2+\lambda^2/a^2} e^{-im2\pi y}\, dy \\ &= -\int_{-\infty}^{\infty} \frac{V}{(y+i\lambda/a)(y-i\lambda/a)} e^{-im2\pi y}\, dy. \end{aligned} \tag{2}$$

When $m \geq 0$, we close the contour in the lower half plane, we get

$$\begin{aligned} V_m &= 2\pi i \frac{1}{a^2} \frac{V}{-2i\lambda/a} e^{-im2\pi(-i\lambda/a)} \\ &= -\frac{\pi V}{a\lambda} e^{-2\pi m\lambda/a}. \end{aligned} \tag{3}$$

When $m < 0$, we close the contour in the upper half plane, we get

$$V_m = -\frac{\pi V}{a\lambda} e^{2\pi m\lambda/a}. \tag{4}$$

Therefore, we have

$$V_m = -\frac{\pi V}{a\lambda} e^{-2\pi|m|\lambda/a}. \tag{5}$$

(c) Typical electron has kinetic energy

$$E \sim \frac{\pi^2\hbar^2}{2ma^2}. \tag{6}$$

For the potential to be weak, we want

$$\frac{\pi V}{a\lambda} \ll \frac{\pi^2\hbar^2}{2ma^2}, \tag{7}$$

which implies

$$V \ll \frac{\pi\lambda\hbar^2}{2ma}. \tag{8}$$

(d) By perturbation theory, the perturbed state is

$$|\tilde{p}\rangle = |p| + \frac{1}{E_p - \hat{H}_0}\hat{V}|p\rangle, \tag{9}$$

where

$$V(x) = \sum_m V_m e^{imkx}. \tag{10}$$

The first three bands emerge at $m = \pm 1, \pm 2, \pm 3$, corresponding to momenta $p = \pm\pi/a, \pm 2\pi/a, \pm 3\pi/a$. The first-order scattering scatters between momenta $m\pi/a$, and $-m\pi/a$. We let

$$|+\rangle = |\frac{m\pi}{a} + \delta\rangle, \quad |-\rangle = |\frac{m\pi}{a} - \delta\rangle. \tag{11}$$

In the subspace spanned by $|+\rangle$ and $|-\rangle$, the Hamiltonian is of the form

$$H_{\text{eff}} = \begin{pmatrix} \epsilon_+ & V_m \\ V_m & \epsilon_- \end{pmatrix}, \tag{12}$$

where

$$\epsilon_\pm = \frac{1}{2m_e}\left(\frac{m\pi}{a}\right)^2 + \frac{\delta^2}{2m_e} \pm \frac{1}{m_e}\frac{m\pi\delta}{a}, \tag{13}$$

so that

$$H_{\text{eff}} = \left(\frac{1}{2m_e}\left(\frac{m\pi}{a}\right)^2 + \frac{\delta^2}{2m_e}\right)\mathbb{1} + \frac{m\pi\delta}{m_e a}\sigma_z + V_m\sigma_x. \tag{14}$$

Taking $\delta \to 0$, we have

$$H_{\text{eff}} = \frac{1}{2m_e}\left(\frac{m\pi}{a}\right)^2\mathbb{1} + V_m\sigma_x, \tag{15}$$

so that

$$E_\pm = \frac{1}{2m_e}\left(\frac{m\pi}{a}\right)^2 \pm |V_m|, \tag{16}$$

and $\Delta E_m = 2V_m$. Therefore, the first band gap emerges at $p = \pm\pi/a$,

$$\Delta E_1 = \frac{2\pi V}{a\lambda}e^{-2\pi\lambda/a}. \tag{17}$$

The second band gap emerges at $p = \pm 2\pi/a$,

$$\Delta E_1 = \frac{2\pi V}{a\lambda}e^{-4\pi\lambda/a}. \tag{18}$$

The third band gap emerges at $p = \pm 3\pi/a$,

$$\Delta E_1 = \frac{2\pi V}{a\lambda}e^{-6\pi\lambda/a}. \tag{19}$$

Problem 2

(a) The condition for constructive interference:

$$
\begin{aligned}
d\sin\alpha &= n\lambda \\
&= n\frac{2\pi}{k} \\
&= n\frac{2\pi\hbar}{p}, \;\; n\in\mathbb{Z},
\end{aligned} \tag{20}
$$

which implies that

$$\sin\alpha = \frac{2n\pi\hbar}{pd}, \;\; n\in\mathbb{Z}, \tag{21}$$

and

$$\alpha = \sin^{-1}\left(\frac{2n\pi\hbar}{pd}\right). \tag{22}$$

(b) The Berry phase is half the solid angle subtended by the spin trajectory. We first consider

$$B_+ = -B_0\left[\hat{z} - r\Big(\hat{x}\cos(2\pi y/L) + \hat{y}\sin(2\pi y/L)\Big)\right], \tag{23}$$

the spin of the electron thus points in the direction $\hat{z} - r\Big(\hat{x}\cos(2\pi y/L) + \hat{y}\sin(2\pi y/L)\Big)$, and it traverses a circle of radius r. For $r \ll 1$, the solid angle subtended by the circle, which is the area of the cap enclosed by the circle, is approximately given by πr^2, because the cap is extremely "flat". Therefore, the Berry phase that a path through the right slit incurs is

$$\theta_+ = \frac{1}{2}\pi r^2. \tag{24}$$

Similarly, the Berry phase that a path through the left slit incurs is

$$\theta_- = -\frac{1}{2}\pi r^2. \tag{25}$$

When the Berry phases are taken into account, constructive interference happens when

$$\frac{2\pi d\sin\alpha}{\lambda} - \pi r^2 = 2\pi n, \quad n\in\mathbb{Z}, \tag{26}$$

and

$$\alpha = \sin^{-1}\left(\frac{2n\pi\hbar}{pd} + \frac{\pi\hbar r^2}{pd}\right) \tag{27}$$

(c) For the adiabatic assumption to apply, we need the rate of change of the ground state to be small compared to the energy gap. Now the energy gap is $|E| \approx 2\mu B_0$, and the rate of change of the ground state is given by

$$\frac{d\psi}{dt} \sim \text{velocity of spin precession} \sim r\frac{2\pi\dot{y}}{L} \approx r\frac{2\pi p}{mL}. \tag{28}$$

Thus

$$|\frac{d\psi}{dt}| \ll |E| \tag{29}$$

implies that

$$B_0 \gg \frac{\pi\hbar p r}{mL\mu}. \tag{30}$$

Problem 3

(a) The energy eigenvalues are given by

$$\pm\epsilon_{\vec{p}} = \pm v\sqrt{p_x^2 + p_y^2}. \tag{31}$$

Thus,

$$\epsilon_{\vec{p}} = v\sqrt{p_x^2 + p_y^2}. \tag{32}$$

(b) The Boltzmann equations reads

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \frac{\partial \vec{r}}{\partial t}\cdot\nabla_{\vec{r}} f + \frac{\partial \vec{p}}{\partial t}\cdot\nabla_{\vec{p}} f = -\frac{1}{\tau}(f - f_0), \tag{33}$$

which implies

$$eE\hat{x}\cdot\nabla_{\vec{p}} f = -\frac{1}{\tau}(f - f_0). \tag{34}$$

Hence

$$f_\pm = f_\pm^{(0)} - \tau e\vec{E}\cdot(\nabla_{\vec{p}} f_\pm). \tag{35}$$

(c) To first order in τ,

$$\begin{aligned}\delta f_\pm(\vec{p}) &= -\tau e\vec{E}\cdot(\nabla_{\vec{p}} f_\pm^{(0)}) \\ &= -\tau eEv\cos\theta\frac{\partial f_\pm^{(0)}}{\partial\epsilon},\end{aligned} \tag{36}$$

where

$$f_\pm^{(0)} = \frac{1}{e^{\pm\epsilon/T}+1}. \tag{37}$$

Thus,

$$\begin{aligned}\delta f_\pm(\vec{p}) &= \tau eE\cos\theta v\frac{e^{\pm\epsilon/\tau}/\tau}{(e^{\pm\epsilon/\tau}+1)^2} \\ &= \tau eEv\frac{\hat{x}\cdot\vec{p}}{|\vec{p}|}\frac{e^{\pm\epsilon/\tau}/\tau}{(e^{\pm\epsilon/\tau}+1)^2}.\end{aligned} \tag{38}$$

(d) The current is given by

$$\vec{j}\cdot\hat{x} = -2e\sum_\pm\int\frac{d^2p}{(2\pi\hbar)^2}\vec{v}\tau eE\cos\theta v\frac{\partial f_\pm^{(0)}}{\partial\epsilon} \tag{39}$$

(e) We have

$$\vec{j} = \sigma\vec{E}, \tag{40}$$

so that

$$\begin{aligned}\sigma &= -2e^2\sum_\pm\int\frac{d^2p}{(2\pi\hbar)^2}v^2\tau\cos^2\theta\frac{\partial f_\pm^{(0)}}{\partial\epsilon} \\ &= -2e^2\sum_\pm\int\frac{d^2p}{(2\pi\hbar)^2}v^2\tau\cos^2\theta\frac{e^{\pm\epsilon/T}/T}{(e^{\pm\epsilon/T}+1)^2}.\end{aligned} \tag{41}$$

Now $d^2p = |p|d|p|d\theta$, $\epsilon = v|p|$, hence $d^2p = (\epsilon/v^2)d\epsilon d\theta$, and

$$\begin{aligned}\sigma &\propto \int d\epsilon d\theta\ \frac{\epsilon e^{\epsilon/T}/T}{(e^{\epsilon/T}+1)^2} \\ &\propto T\int d(\epsilon/T)d\theta \frac{e^{\epsilon/T}(\epsilon/T)}{(e^{\epsilon/T}+1)^2} \\ &\propto T, \end{aligned} \tag{42}$$

so $\gamma = 1$.

Problem 4

(a) The equations of motion are given by

$$\dot{\vec{p}} = -e\vec{E} = -eE\hat{x}, \tag{43}$$

and

$$\begin{aligned}\dot{\vec{r}} &= \nabla_{\vec{p}}\epsilon_{\vec{p}} + \dot{\vec{p}} \times \vec{\Omega}_{\vec{p}} \\ &= m\Big(\sin p_x\hat{x} + \sin p_y\hat{y}\Big) + \frac{eE}{2\pi}\hat{y}.\end{aligned} \tag{44}$$

(b) Solving (43) with $\vec{p}(0) = 0$ gives

$$\vec{p}(t) = -eEt\hat{x}, \tag{45}$$

so

$$\dot{\vec{r}} = m\sin(-eEt)\hat{x} + \frac{eE}{2\pi}\hat{y}, \tag{46}$$

which gives

$$\vec{r}(t) = \vec{r}(0) + \frac{m}{eE}\cos(eEt)\hat{x} + \frac{eEt}{2\pi}\hat{y}. \tag{47}$$