

I. REVIEW OF ELECTRONS IN JELLIUM

A. Fermi-Dirac distribution

How are these states get occupied? That depends on the temperature and chemical potential of the electrons. For each state \vec{k} , with a given spin state $\sigma^z = \pm 1$ for electrons, there is a grand-canonical partition function:

$$\mathcal{Z}_{\vec{k}} = \sum_{N_{\vec{k}}=0}^{N_{max}} e^{-(\epsilon_{\vec{k}}-\mu)\beta} = 1 + e^{-(\epsilon_{\vec{k}}-\mu)/T} \quad (1)$$

since for Fermions, each state can be either occupied or empty.

The occupation follows very quickly. For every state we have:

$$n_{\epsilon_{\vec{k}}} = \frac{e^{-(\epsilon_{\vec{k}}-\mu)/T}}{1 + e^{-(\epsilon_{\vec{k}}-\mu)/T}} \quad (2)$$

and the total number:

$$N = V \int d\epsilon \rho(\epsilon) n_{\epsilon} = V(2s+1) \int \frac{d^d k}{(2\pi)^d} \frac{1}{e^{(\epsilon_{\vec{k}}-\mu)/T} + 1} \quad (3)$$

This is a very important result. Let's interpret it. Everything in front of the fraction is simply an enumeration of the states - $d^d k/(2\pi)^d$, $(2s+1)V$, it's all counting stuff. The integrand, $n_{\epsilon_{\vec{k}}} = f(\epsilon_{\vec{k}})$, is called the Fermi-function, or the Fermi-Dirac distribution.

The Fermi-Dirac distributions for the theory of metals is what the cosine function is to the theory of triangles. With it, you can calculate everything. And like knowing the cosine at small angles, you can get away with a lot just knowing how the FD distribution behaves at a low temperature.

What does this function look like at zero temperature? Assuming that our energy spectrum stretches from zero up, (which we can always do by setting the lowest energy eigenvalue in the problem to zero), if $\mu < 0$ then:

$$f(\epsilon_{\vec{k}}) = \frac{1}{e^{(\epsilon_{\vec{k}}-\mu)/T} + 1} \rightarrow 0 \quad (4)$$

since the exponent is large. For positive μ it becomes:

$$f(\epsilon_{\vec{k}}) = \frac{1}{e^{(\epsilon_{\vec{k}}-\mu)/T} + 1} = \begin{cases} 1 & \epsilon_{\vec{k}} < \mu \\ 0 & \epsilon_{\vec{k}} > \mu \end{cases} \quad (5)$$

It is a step like function. Only states below the chemical potential are occupied, we call the chemical potential in this case - the Fermi energy.

Let's calculate it. We equate the number of states up to the fermi energy with the number of particles in the system.

$$n = \int_0^{E_F} d\epsilon \rho(\epsilon) \quad (6)$$

This should be considered as the definition of the Fermi energy. Literally, we take the convolution of the density of states, and the fermi-dirac distribution, and integrate until we managed to fit all the particles in the system into a quantum state.

In the case of a spherically-symmetric dispersion, we can write the k-integral very simply:

$$n = (2s+1) \int_0^{k_F} \frac{d^d k}{(2\pi)^d} = (2s+1) \Omega_d \frac{1}{(2\pi)^d} \frac{k_F^d}{d} = (2s+1) \frac{K}{(2\pi)^d} \quad (7)$$

where K is the volume inside the Fermi-sphere.

As a quick example let's do this in 2d. It is fun to do this in k-space:

$$n = \frac{(2s+1)}{4\pi} k_F^2 \quad (8)$$

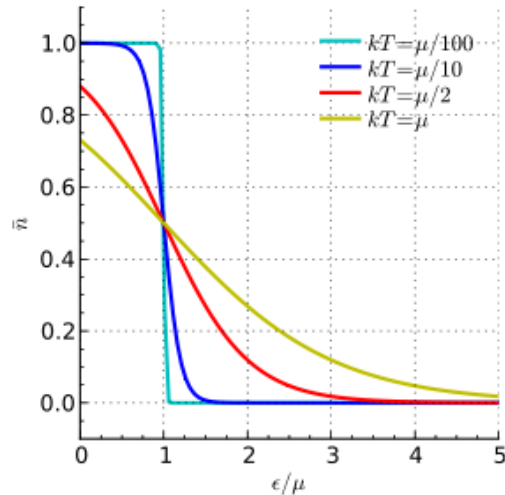


FIG. 1. The Fermi-Dirac distribution as a function of energy. Taken from wikipedia.

We can of-course write this in terms of energy:

$$= \frac{(2s+1) 2mE_F}{4\pi \hbar^2} \quad (9)$$

This defines the fermi energy, but the cool thing lies in the k-space determination: upto a factor of π and the spin-related degeneracy, the fermi wave number, k_F , is just the inverse interparticle distance.

In a metal, usually there is just one particle per atom. The distance between atoms is of the order of 5\AA . What is the Fermi energy (assume 3d metal)?

$$n = 2 \int_0^{k_F} \frac{d^3k}{(2\pi)^3} = \frac{1}{(2\pi)^3} \frac{4\pi}{3} k_F^3 \cdot 2 \approx \frac{1}{\pi^3} k_F^3 \quad (10)$$

and using the famous near-law of the state of Indiana: $\pi = 3$. Actually, it was $\pi = 3.2$, and the bill almost got through in the year of 1897. It passed a first reading, but not a second one. The final result is then:

$$k_F = \pi \frac{1}{a} = \frac{\pi}{5 \cdot 10^{-10} m} \quad (11)$$

and energy:

$$E_F = \frac{\hbar^2 \pi^2}{2 \cdot 10^{-30} kg (5 \cdot 10^{-10} m)^2} \approx 2 \cdot 10^{-19} J = 1.25 eV = 1.5 \cdot 10^4 K. \quad (12)$$

Indeed the fermi energy of most metals is very high, and is of the order of between 10^4 and 10^5 K. A more amazing number is the fermi velocity:

$$v_F = \frac{\hbar k_F}{m} = \frac{\pi \hbar}{ma} \approx 7 \cdot 10^5 m/s \quad (13)$$

Damn fast for something that is actually at zero temperature!

II. PHYSICAL PROPERTIES OF A DEGENERATE ELECTRON GAS

Last lecture we derived the Fermi-Dirac distribution and just barely started calculating with it. Next, we derive the main physical properties of an electron gas at low temperature. We will calculate the zero-temperature degeneracy pressure, and zero-temperature magnetic susceptibility, and also the heat capacity at low temperatures. The thing that we will see over and over again, is that the properties of the electron gas are completely determined by what's going on near the fermi surface. Everything else is frozen out. This makes the physics of electrons quite intuitive - please keep your eyes peeled, and see how this occurs.

A. Example: Electronic degeneracy pressure

As a first example for the zero-temperature Fermi-Dirac distribution, we calculate the zero-temperature degeneracy pressure. First, recall from your stat. mech. classes that the grand canonical ensemble, is connected to the grand canonical potential, which obeys:

$$-T \ln \mathcal{Z} = \Omega(V, \mu, T) = -pV \quad (14)$$

Armed with this, let's concentrate on 2d:

$$-pV = -2TV \int_0^\infty \frac{d^2k}{(2\pi)^2} \ln(e^{-(\epsilon_{\vec{k}} - E_F)/T} + 1) \quad (15)$$

the argument of the log should not mislead you - when $E > E_F$ it is 1, yielding zero for the integrand. When $E < E_F$, the exponent dominates, and we need to ignore the one, the final expression is just:

$$= -2TV \int_0^{k_F} \frac{d^2k}{(2\pi)^2} \frac{(\epsilon_{\vec{k}} - E_F)}{T} = -N(E_F - \langle \epsilon \rangle) = -\frac{1}{2}NE_F \quad (16)$$

Finally:

$$p = \frac{1}{2}n^2 \frac{\hbar^2}{4\pi m} \quad (17)$$

independent of temperature, and grows quadratically with density. This can be thought of as degeneracy pressure - the electrons would like to have absolutely no kinetic energy at zero temperature, but they can't quite do it. Instead, they press each other to higher energy levels, and hence this zero temperature pressure. In real numbers for a typical metal: this comes out to be about 50000 times atmospheric pressure!