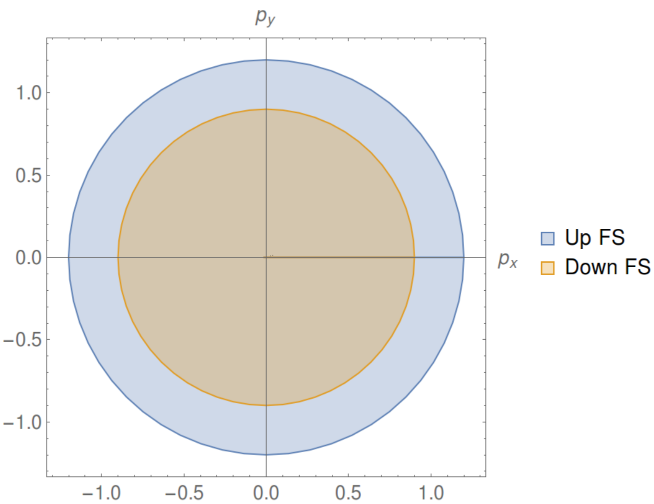

FIG. 1. A magnetic field splits the Fermi surfaces of the two spins, and there are more spins pointing along the field than against it.The electrons on the outer annulus give rise to a net spin polarization and magnetization.

Oct 7th - Class 3, Ph 135, Gil Refael

I. MAGNETIC PROPERTIES OF FREE ELECTRONS

A. Pauli Susceptibility of an electron gas

Another important property of the zero-temperature electron gasses is the Pauli-spin-Susceptibility. When we turn on a magnetic field, the spin Hamiltonian gets out of deep freeze, and we need to put a term for it in the hamiltonian:

$$\hat{\mathcal{H}}_{spin} = -\frac{g}{2}\mu_B \vec{H}\cdot\hat{\vec{\sigma}} = -\frac{g}{2}\mu_B H\sigma_z \tag{1}$$

choosing the spin-quantization access along the magnetic field. Now, spin up has less energy then spin down. The chemical potential of the electron gas is still the same for the two spins, but this implies that the fermi energy - which here we distinguish from the chemical potential by saying that it is the *kinetic* energy upto which electron states are filled - is not the same for the two electron flavors.

For down electrons, the total energy is:

$$E_\downarrow = \epsilon_{\vec{k}} + \frac{1}{2}Hg\mu_B \tag{2}$$

while

$$E_\uparrow = \epsilon_{\vec{k}} - \frac{1}{2}Hg\mu_B \tag{3}$$

Since the energy shift for up is minus that of down, some electrons from the high-energy down spins, would flip to point in the direction of the magnetic field. This will happen until we have

$$E_{F\uparrow} - \frac{1}{2}Hg\mu_B = E_{F\downarrow} + \frac{1}{2}Hg\mu_B \tag{4}$$

So, as you can see from Fig. 1, we have an excess of up spins, and we can easily calculate how many:

$$n_\uparrow - n_\downarrow = \int\limits_{\mu - Hg\mu_B/2}^{\mu + Hg\mu_B/2} \rho(E)dE = \rho(E_F)Hg\mu_B \tag{5}$$

Where $\rho(E)$ here is the single-spin DOS. The excess spin magnetization is then:

$$\frac{1}{2} g\mu_B (n_\uparrow - n_\downarrow) = \frac{1}{2}\rho(E_F) H (g\mu_B)^2 \tag{6}$$

So the susceptibility is:

$$\chi_{Pauli} = \frac{1}{2}\rho(E_F)(g\mu_B)^2 \tag{7}$$

It is a direct measure of the density of states at the Fermi surface. Free electron spins tend to point in the direction of the field, but don't actually have magnetization in zero field. Hence this is called **paramagnetism**.

Now, in addition to this effect there is also going to be an orbital effect on the general magnetization, and the question arises - how do we measure the Pauli susceptibility alone? This is done using the Knight shift - The energy level of nuclear spins are coupled to the electronic spins through the contact term of the hyper-fine hamiltonian of an atom. This coupling is much stronger than the coupling of nuclei to a raw magnetic field (since their Bohr magneton is three orders of magnitude smaller than that of the electron's). Using NMR one can get very accurate readings of nuclear energy levels, and there one see's precisely this quantity.

B. Orbital effects of a magnetic field - Shubnikov-De-Haas-van Alpen oscillations and Landau diamagnetism

A magnetic field also has a huge orbital effect on electronic motion. From classical mechanics, we know that an electron in a magnetic field starts to move in circles Newton's third law is:

$$mv^2/R = evB \tag{8}$$

and:

$$\frac{v}{R} = \omega_c = \frac{eB}{m} \tag{9}$$

If we look at our electron gas, in 2-d the entire fermi sphere essentially rotates with this cyclotron angular frequency. Why? becuase the velocity vector of the particle in a field carries out basically the same rotation as the location relative to the center of the circle, only $\pi/2$ behind.

This semi-classical picture is true for small fields, but with slightly larger fields, Quantum mechanics wakes up with a vengence, and dictates - there is a distinct frequency in the problem. Therefore the energies must be quantized into multiples of that frequency. And thus we get Landau-levels. Like in an Harmonic oscillator, the allowed energy eigenvalues of a particle in a magnetic field are:

$$E_n = \left(\frac{1}{2} + n\right)\hbar\omega_c \tag{10}$$

In two dimensions, instead of having a flat density of states, we suddenly have a comb like shape with spikes at these energies.

But how high are these spikes? The answer, which I will state but not derive. As you may know when you let an electron go in a closed orbit, the magnetic flux inside the circle becomes a modular variable. The effect of the field on the electron's wave-function phase is:

$$\varphi \sim \frac{e}{\hbar}\int d\vec{l}\cdot\vec{A} = 2\pi\frac{\Phi}{h/e} \tag{11}$$

Whenever φ is a multiple of 2π, the electron can be at rest. This gives rise to a flux quantum, the fluxon:

$$\Phi_0 = \frac{h}{e} \approx 4\cdot 10^{-15} T\cdot m^2 \tag{12}$$

Note that flux quantization depends very strongly on the charge of the elementary particle you are considering.

A magnetic field is like a flux density. The most useful way is to think about it as the field for a given 2d fluxon density:

$$B = \frac{\Phi_0}{b}, \tag{13}$$

with b the area that corresponds to a flux of a single fluxon. As it turns out, for each landau level, there is a density of states which is exactly $1/b$ - just as though there is one electronic state for each fluxon. This means that the density of states function for an electron gas in 2d is (ignoring spin) :

$$\rho(\epsilon) = \sum_{n=0}^{\infty} \delta(\epsilon - (\frac{1}{2} + n)\hbar\omega_c)\frac{B}{\Phi_0} \tag{14}$$

The integer quatum hall effect occurs when the number of electrons fits exactly an integer number of landau levels. We define the filling factor:

$$\nu = \frac{n}{B/\Phi_0}. \tag{15}$$

ν tells us how many landau levels are full. The Qauntum-Hall effect occurs when ν hits the magic numbers $1, 2, 3\ldots$ for integer, and a variety of rational fractions - most notably 1/3, for which Laughlin Stoermer and Tsui won their Nobel Prize. In these phases the 2d electron gas becomes gapped, and its Hall conductance is exactly quantized. (The Hall conductance is the current response to a perpendicular field: $j_x = \sigma_{xy}E_y$, for instance. The quantization is to $\sigma_{xy} = \nu\frac{e^2}{h}$, and we'll see how it arises even in this quarter.) These regimes of small ν, i.e., of high field, are complex, and have a lot of structure, which , is in the scope of ph223. We will concentrate on the relatively simple, but nevertheless fascinating phenomena at low fields.

As you saw in the case of the Pauli susceptibility, everything happens near the edge of the fermi surface. Looking at the 2d electron gas, which is fondly called 2DEG, when we turn on a magnetic field, the electronic states just bunch into delta-functions. Big deal - all those states that are filled - instead of being a continuum, they are just bunched together. But they are frozen. even if we turn on a spin-field, as in the Pauli susceptibility, the depolarization occurs at the Fermi-surface - not deep inside the fermi sphere. When we heat up the gas a bit, the fermi distribution softens up here, but not further than a temperatrue T within the fermi sphere. But near the fermi surface, the thing that really matters once a field is on, is how filled the last Landau level is. This is:

$$\nu mod 1 \tag{16}$$

Every quantity of the system will depend strongly on $\nu mod(1)$, but only weakly on what ν is, or on the magnetic field. Thus, every quantity will undergo an oscillation, whenever ν goes through 1. But This means:

$$\nu = \frac{\frac{1}{(2\pi)^2}K}{B/\Phi_0} \tag{17}$$

Where K is the volume of the fermi-sphere in k-space. So as a function of $1/B$ we'll get oscillations that are with 'periodicity':

$$\Delta(1/B) = \frac{(2\pi)^2}{K\Phi_0} \tag{18}$$

which corresponds to $1/B$ changes that move ν to $\nu - 1$.

These oscillations are called Shubnikov - De-Haas (condcutivity oscillations), and De-Haas - Van-Alpen (susceptibility) oscillations. This result is remarkable: if you measure the resistance of a metal in a varying field, plot the result with respect to $1/B$, you will see oscillations whose frequency corresponds to the area enclosed in the fermi surface (and I'm careful not to say Fermi-sphere, because the shape may not be a sphere- that depends on the hamiltonian). If we carry this out in 3d, then instead of K we get:

$$\Delta(1/B) = \frac{(2\pi)^2}{K_\perp\Phi_0} \tag{19}$$

where $K_\perp$ is the area of the projected fermi-sphere. By measuring this in various magnetic field directions, we get a wealth of information about the shape of the fermi-surface.

There's another thing. Because the dispersion of the electrons changes, so does their energy. In fact it rises:

$$E = E_0 + \frac{1}{2}\chi_L B^2 \tag{20}$$

The extra kinetic energy due to the magnetic field means also that the electrons, in their orbits, produce a magnetization contrary to the external field and try to reduce it. This contribution to the magnetization is called Landau-diamagnetism. In the problem set you'll find:

$$\chi_L = -\frac{1}{3}\chi_{Pauli} \tag{21}$$

in a free electron gas.

A bit of terminology - free spins tend to point at the direction of the external field, but not to order ferromagnetically on their own. This is paramagnetism, and it implies a negative magnetic energy. Diamagnetism is the opposite. The magnetic moment tries to reduce the external field. Another form of magnetism is antiferromagnetism, where the spins make a spatial pattern with nearest spin pointing opposite to eashother, and therefore their magnetic moments cancel. The difference between diamagnetism and antiferromagnetic behavior is that without a field there isn't any spontaneous canted internal field.