I. ORBITAL EFFECTS OF A MAGNETIC FIELD - SHUBNIKOV-DE-HAAS-VAN ALPEN OSCILLATIONS AND LANDAU DIAMAGNETISM

A magnetic field also has a huge orbital effect on electronic motion. From classical mechanics, we know that an electron in a magnetic field starts to move in circles Newton's third law is:

$$mv^2/R = evB \tag{1}$$

and:

$$\frac{v}{R} = \omega_c = \frac{eB}{m} \tag{2}$$

If we look at our electron gas, in 2-d the entire fermi sphere essentially rotates with this cyclotron angular frequency. Why? becuase the velocity vector of the particle in a field carries out basically the same rotation as the location relative to the center of the circle, only $\pi/2$ behind.

This semi-classical picutre is true for small fields, but with slightly larger fields, Quantum mechanics wakes up with a vengence, and dictates - there is a distinct frequency in the problem. Therefore the energies must be quantized into multiples of that frequency. And thus we get Landau-levels. Like in an Harmonic oscillator, the allowed energy eigenvalues of a particle in a magnetic field are:

$$E_n = \left(\frac{1}{2} + n\right)\hbar\omega_c \tag{3}$$

In two dimensions, instead of having a flat density of states, we suddenly have a comb like shape with spikes at these energies.

But how high are these spikes? The answer, which I will state but not derive. As you may know when you let an electron go in a closed orbit, (DRAW AN ELECTRON ON A RING) the magnetic flux inside the circle becomes a modular variable. The effect of the field on the electron's wave-function phase is:

$$\varphi \sim \frac{e}{\hbar}\int d\vec{l}\cdot\vec{A} = 2\pi\frac{\Phi}{h/e} \tag{4}$$

Whenever φ is a multiple of 2π, the electron can be at rest. This gives rise to a flux quantum, the fluxon:

$$\Phi_0 = \frac{h}{e} \tag{5}$$

Note that flux quantization depends very strongly on the charge of the elementary particle you are considering.

A magnetic field is like a flux density. The mmost useful way is to think about it as the field for a given 2d fluxon density:

$$B = \frac{\Phi_0}{b} \tag{6}$$

as it turns out, for each landau level, there is a density of states which is exactly b - just as though there is one electronic state for each fluxon. This means that the density of states function for an electron gas in 2d is (ignoring spin) :

$$\rho(\epsilon) = \sum_{n=0}^{\infty}\delta(\epsilon - (\frac{1}{2}+n)\hbar\omega_c)\frac{B}{\Phi_0} \tag{7}$$

The integer quatum hall effect occurs when the number of electrons fits exactly an integer number of landau levels. We define the filling factor:

$$\nu = \frac{n}{B/\Phi_0} \tag{8}$$

ν tells us how many landau levels are full. The Qauntum-Hall effects occur when ν hits the magic numbers $1, 2, 3\ldots$ for integer, and a variety of rational fractions - most notably $1/3$, for which Laughlin Stoermer and Tsui won their nobel prize for.

But those regime of small ν, ie, of high field, they are complex, and have a lot of structure, which , unfortunately, lies beyond the scope of this class. We will concentrate on the relatively simple, but nevertheless fascinating phenomena at low fields.

As you saw in the case of the Pauli susceptibility, everything happens near the edge of the fermi surface. Looking at the 2d electron gas, which is fondly called 2DEG, when we turn on a magnetic field, the electronic states just bunch into delta-functions. Big deal - all those states that are filled - instead of being a continuum, they are just bunched together. But they are frozen. even if we turn on a spin-field, as in the Pauli susceptibility, the depolarization occurs HERE (SHOW Fermi-surface on board). Not deep inside the fermi sphere. When we heat up the gas a bit, the fermi distribution softens up here, but not further than a temperatrue T within the fermi sphere. But near the fermi surface - the thing that really maters once a field is on, is how filled the last landau level is. This is:

$$mod(\nu) \tag{9}$$

Every quantity of the system will depend strongly on $mod(\nu)$, but only weakly on what ν is, or on the magnetic field. Thus, every quantity will undergo an oscillation, whenever ν goes through 1. But This means:

$$\nu = \frac{\frac{1}{(2\pi)^2}K}{B/\Phi_0} \tag{10}$$

Where K is the volume of the fermi-sphere in k-space. So as a function of $1/B$ we'll get oscillations that are with 'wave-length':

$$\Delta(1/B) = \frac{(2\pi)^2}{K\Phi_0} \tag{11}$$

which corresponds to $1/B$ changes that move ν to $\nu - 1$.

These oscillations are called Shubnikov-De-Haas-Van-Alpen oscillations. This result is remarkable - if you measure the resistance of a metal in a varying field, plot the result with respect to $1/B$, we'll see oscillations, whose frequency, corresponds to the volume enclosed in the fermi surface (and I'm careful not to say Fermi-sphere, because the shape may not be a sphere- that depends on the hamiltonian). If we carry this out in 3d, then instead of K we get:

$$\Delta(1/B) = \frac{(2\pi)^2}{K_\perp\Phi_0} \tag{12}$$

where $K_\perp$ is the area of the projected fermi-sphere. By measuring this in various magnetic field directions, we get a wealth of information about the shape of the fermi-surface.

There's another thing. Because the dispersion of the electrons changes, so does their energy. In fact it rises:

$$E = E_0 + \frac{1}{2}\chi_L B^2 \tag{13}$$

The extra kinetic energy due to the magnetic field means also that the electrons, in their orbits, produce a magnetization contrary to the external field and try to reduce it. This contribution to the magnetization is called Landau-diamagnetism. In the problem set you'll find:

$$\chi_L = -\frac{1}{3}\chi_{Pauli} \tag{14}$$

in a free electron gas.

A bit of terminology - free spins tend to point at the direction of the external field, but not to eorder ferromagnetically on their own - this is paramagnetism, and it implies a negative magnetic energy. Diamagnetism is the opposite - the magnetic moment tries to reduce the external field - which is diamagnetism. The difference between diamagnetism and AFM is that without a field there isn't any spontaneous canted internal field.

A. Sommerfeld expansion

The zero temperature properties of the fermi-dirac allowed us to calculate the pressure, Fermi-energy, and the Pauli susceptibility, as well as the Shubnikov de-Haas oscillations, in principle, we could also calculate the Landau diamagnetism, but we left it for later. But how do things change with temperature? How does temperature come in? For this part of class we turn the magnetic field off, and concentrate on temperature effects.

Let's start with a simple quantity - the heat capacity. What would be its dependence on temperature? Before doing a hard-core calculation, let's use the intuition from the Fermi-Dirac ditribution to find it.

If we have a low temperature -

$$T \ll E_F \tag{15}$$

then the fermi-Dirac distribution is 1 or zero outside a region were:

$$e^{(\epsilon_{\vec{k}}-\mu)/T} \tag{16}$$

is not zero or infinity. This region is of thickness $2T$. This is where everything happens at finite temperature. So what is the temperature dependent energy?

At zero temperature we know what the energy is:

$$U = \int_0^{E_F} d\epsilon \rho(\epsilon)\epsilon \tag{17}$$

But this must have low T corrections, which we can presumably calculate as a power series in T. I want ot show you how to get the first corrections simply on the basis of geometry and intution.

We can think of the distribution as essentially a straight line going from zero at $E_F + T$ to 1 at $E_F - T$. So half of the states up here (DRAW THE triangles and the shifted mass) move to higher energy. How many states are we talking about?

$$\overline{n} \approx \rho(E_F)T/4 \tag{18}$$

The energy gain is twice the average energy in each triangle. The center of mass of a triangle is at a third of its height. here it means that the left triangle is $T/3$ away from E_F, and after it moves to the other side, it is displaced, on average the same amount to the right. Thus the energy contribution is:

$$\Delta U = \frac{2}{3}\overline{n}T = \frac{1}{6}\rho(E_F)T^2 \tag{19}$$

This is a T^2 correction to the energy of the degenerate fermi-gas. In fact, we can continue this way, and it turns out that the heat capacity will only have even power of T.

The heat capacity at constant volume is just:

$$\frac{dU}{dT} \sim \rho(E_F)T \tag{20}$$

This is one of the earmarks of an electron gas in metals, and of fermi-liquids in general.

We can't really trust the coefficient completely, but we get the correct temperture dependence.

Now that we got more of a feelign for the fermi dirac distribution, let's derive a systematic way to produce the temperature dependence of quantities relating to fermi gasses. This is the Sommerfeld approximation.

What are the interesting TD quantities for an electron gas?

The energy integral for instance, is:

$$E = V\int_0^{\infty} d\epsilon \rho(\epsilon)\frac{\epsilon}{e^{(\epsilon-\mu)/T}+1} \tag{21}$$

The number integral is:

$$N = V\int_0^{\infty} d\epsilon \rho(\epsilon)\frac{1}{e^{(\epsilon-\mu)/T}+1} \tag{22}$$

The general thing seems to be:

$$G/V = \int_0^{\infty} d\epsilon g(\epsilon)\frac{1}{e^{(\epsilon-\mu)/T}+1} \tag{23}$$

We would like to do to this integral analytically, what we just did to it graphically above, with the heat capacity. The idea is that we think of the FD distribution as a step, and then go back to reconsider:

$$G/V = \int_0^\mu d\epsilon g(\epsilon) + \int_0^\mu d\epsilon g(\epsilon)\left(\frac{1}{e^{(\epsilon-\mu)/T}+1} - 1\right) + \int_\mu^\infty d\epsilon \rho(\epsilon)\frac{1}{e^{(\epsilon-\mu)/T}+1} \tag{24}$$

So we separated the inegral into three parts - the integral upto the chemical potential assuming occupation 1. a correction for the energy parts that are from small energies, and correction from higher energies, suppressed by the distribution. We treat first the second part - carrying out the algebra in the brackets, it is just:

$$\left(\frac{1}{e^{(\epsilon-\mu)/T}+1} - 1\right) = -\frac{1}{e^{-(\epsilon-\mu)\beta}+1} \tag{25}$$

The other thing is - we change the integration limit such that instead of being integrated from zero to μ, we kinda cheat and allow integration from $-\infty$ to μ. This is inly going to hurt us exponentially if at all.

The next steps are: change the integration variables so that they are relative to μ: $\epsilon \to \mu + \epsilon$. This means:

$$G/V = G_0/V - \int_{-\infty}^0 d\epsilon g(\mu+\epsilon)\frac{1}{e^{-\epsilon/T}+1} + \int_0^\infty d\epsilon \rho(\mu+\epsilon)\frac{1}{e^{\epsilon/T}+1} \tag{26}$$

Next, we change the integration variable in the first integral to minus itself: $\epsilon \to -\epsilon$. You can see why - right? The expressions are just very similar on these two. This is the symmetric property of the FD distribution - take off 1, and then change right to left, and you got the same thing. We get:

$$\to G_0/V - \int_0^\infty d\epsilon g(\mu-\epsilon)\frac{1}{e^{\epsilon/T}+1} + \int_0^\infty d\epsilon \rho(\mu+\epsilon)\frac{1}{e^{\epsilon/T}+1} = G_0/V + \int_0^\infty d\epsilon \left(g(\mu+\epsilon) - g(\mu-\epsilon)\right)\frac{1}{e^{\epsilon/T}+1} \tag{27}$$

The next step is to expend the function $g(\epsilon)$ in a taylor series. Only odd powers of ϵ seem to survive:

$$g(\mu+\epsilon) - g(\mu-\epsilon) = 2(g'(\mu)\epsilon + \frac{1}{3!}g'''(\mu)\epsilon^3 \ldots) = 2\sum_{n=1(odd)}^{\infty}\frac{1}{n!}g^{(n)}(\mu)\epsilon^n \tag{28}$$

To finish the expansion we consider a general term in the integral. Once we put the taylor expansion in we would get:

$$g^{(n)}(\mu)\int_0^\infty d\epsilon \epsilon^n \frac{1}{e^{\epsilon/T}+1} \tag{29}$$

Here, we really want the temperature dependence. So we rescale, $\epsilon = xT$:

$$\to g^{(n)}(\mu)T^{n+1}\int_0^\infty dx \frac{x^n}{e^x+1} \tag{30}$$

This is what I wanted - the integral now only contains numbers, and is just a numeber. we can get a series expansion for it pretty easily by expanding the denominator in the exponent:

$$f_n = \int_0^\infty dx \frac{x^n}{e^x+1} = \int_0^\infty dx x^n e^{-x}(1 - e^{-x} + e^{-2x} - e^{-3x}\ldots) = n!\left(1 - \frac{1}{2^{n+1}} + \frac{1}{3^{n+1}}\ldots\right) \tag{31}$$

But the general expression is now:

$$G_0/V + \sum_{n>0,odd} T^{n+1}\frac{2}{n!}g^{(n)}(\mu)f_n \tag{32}$$

very simple...

For the energy we get -

$$E/V = E_0/V + 2T^2 \frac{d}{d\epsilon}(\epsilon\rho(\epsilon))\Big|_{\epsilon=\mu} f_1 = E_0/V + 2f_1T^2(\rho(\mu) + \mu\rho'(\mu)) \tag{33}$$

same T dependence as we found, but not quite the same number.

That's it for Fermions. Fascinating particles. You try to cool them down to absolute zero, and they move at speeds that makes your ferarri blush. you try to confine them into a block of iron, and they press against the walls of their confinement with pressure that would crush us to little crums. They dictate the behavior of cold metals, and that of extremely hot white dwarves. amazing.